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Emergent spacetime and the origin of gravity

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Abstract: We present an exposition on the geometrization of the electromagnetic force. We show that, in noncommutative (NC) spacetime, there always exists a coordinate transformation to locally eliminate the electromagnetic force, which is precisely the Darboux theorem in symplectic geometry. As a consequence, the electromagnetism can be realized as a geometrical property of spacetime like gravity. We show that the geometrization of the electromagnetic force in NC spacetime is the origin of gravity, dubbed as the emergent gravity. We discuss how the emergent gravity reveals a novel, radically different picture about the origin of spacetime. In particular, the emergent gravity naturally explains the dynamical origin of flat spacetime, which is absent in Einstein gravity. This spacetime picture turns out to be crucial for a tenable solution of the cosmological constant problem.

Keywords: Models of Quantum Gravity, Gauge-gravity correspondence, Non-Commutative Geometry

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Contents

1 Deformation theory

One of the main trends in modern physics and mathematics is to study a theory of deformations. Deformations are performed first to specify a particular structure (e.g., complex, symplectic, or algebraic structures) which one wants to deform, and then to introduce a deformation parameter $[\hbar]$ such that the limit $[\hbar] \to 0$ recovers its parent theory. The most salient examples of the deformation theories are Kodaira-Spencer theory, deformation quantization, quantum group, etc. in mathematics and quantum mechanics, string theory, noncommutative (NC) field theory, etc. in physics. Interestingly, consequences after the deformation are often radical: A theory with $[\hbar] \neq 0$ is often qualitatively different from its parent theory and reveals a unification of physical or mathematical structures (e.g., wave-particle duality, mirror symmetry, etc.).

Let us focus on the deformation theories appearing in physics. Our mission is to deform some structures of a point-particle theory in classical mechanics. There could be several in general, but the most salient ones among them are quantum mechanics, string theory and NC field theory, which we call $\hbar$-deformation, α'-deformation and θ-deformation, respectively. The deformation parameter $[\hbar]$ (which denotes a generic one) is mostly a dimensionful constant and plays a role of a conversion factor bridging two different quantities, e.g., $p = 2\pi\hbar/\lambda$ for the famous wave-particle duality in quantum mechanics. The introduction of the new constant $[\hbar]$ into the theory is not a simple addition but often a radical change of the parent theory triggering a new physics. Let us reflect the new physics sprouted up from the $[\hbar]$-deformation, which never exists in the $[\hbar] = 0$ theory.

Quantum mechanics is the formulation of mechanics in NC phase space

$$[x^i, p_k] = i\hbar\delta^i_k. \tag{1.1}$$

The deformation parameter $\hbar$ is to deform a commutative Poisson algebra of observables in phase space into NC one. This $\hbar$-deformation (quantum mechanics) has activated revolutionary changes of classical physics. One of the most prominent physics is the wave-particle duality whose striking physics could be embodied in the two-slit experiment.

String theory can be regarded as a deformation of point-particle theory in the sense that zero-dimensional point particles are replaced by one-dimensional extended objects, strings, whose size is characterized by the parameter α'. This α'-deformation also results in a fundamental change of physics, which has never been observed in a particle theory. It is rather a theory of gravity (or grandiloquently a theory of everything). One of the striking consequences due to the α'-deformation is 'T-duality', which is a symmetry between small and large distances, symbolically represented by

$$R \leftrightarrow \frac{\alpha'}{R}. \tag{1.2}$$

The T-duality is a crucial ingredient for various string dualities and mirror symmetry.

NC field theory is the formulation of field theory in NC spacetime

$$[y^a, y^b]_\star = i\theta^{ab}. \tag{1.3}$$

See [1, 2] for a review of this subject. We will consider only space-noncommutativity throughout the paper in spite of the abuse of the term 'NC spacetime' and argue in section 4.1 that "Time" emerges in a different way. This NC spacetime arises from introducing a symplectic structure $B = \frac{1}{2}B_{ab}dy^a \wedge dy^b$ and then quantizing the spacetime with its Poisson structure $\theta^{ab} \equiv (B^{-1})^{ab}$, treating it as a quantum phase space. In other words, the spacetime (1.3) becomes a NC phase space. Therefore the NC field theory, which we call θ-deformation, is mathematically very similar to quantum mechanics. They are all involved with a NC $\star$-algebra generated by eq. (1.1) or eq. (1.3). Indeed we will find many parallels. Another naive observation is that the θ-deformation (NC field theory) would be

Theory	Deformation	New physics
Quantum mechanics	$\hbar$	wave-particle duality
String theory	α'	T-duality
NC field theory	θ^{ab}	?

Table 1. $[\hbar]$-deformations and their new physics.

much similar to the α'-deformation from the viewpoint of deformation theory since the deformation parameters α' and θ equally carry the dimension of (length)^2. A difference is that the θ-deformation is done in the field theory framework. We will further elaborate the similarity in this paper.

What is a new physics due to the θ-deformation ? A remarkable fact is that translations in NC directions are an inner automorphism of NC $\star$-algebra $\mathcal{A}_\theta$, i.e., $e^{ik\cdot y} \star \widehat{f}(y) \star e^{-ik\cdot y} = \widehat{f}(y + \theta \cdot k)$ for any $\widehat{f}(y) \in \mathcal{A}_\theta$ or, in its infinitesimal form,

$$[y^a, \widehat{f}(y)]_\star = i\theta^{ab}\partial_b \widehat{f}(y). \tag{1.4}$$

In this paper we will denote NC fields (or variables) with the hat as in eq. (1.4) but we will omit the hat for NC coordinates y^a in eq. (1.3) for notational convenience. We will show that the θ-deformation is seeding in it the physics of the α'-deformation as well as the $\hbar$-deformation, so to answer the question in the table 1.

This paper is organized as follows. In section 2 we review the picture of emergent gravity presented in [3] with a few refinements. First we consolidate some results well-known from string theory to explain why there always exists a coordinate transformation to locally eliminate the electromagnetic force as long as D-brane worldvolume M supports a symplectic structure B, i.e., M becomes a NC space. That is, the NC spacetime admits a novel form of the equivalence principle, known as the Darboux theorem, for the geometrization of the electromagnetism. It turns out [3] that the Darboux theorem as the equivalence principle in symplectic geometry is the essence of emergent gravity. See the table 2. In addition we add a new observation that the geometrization of the electromagnetism in the B-field background can be nicely understood in terms of the generalized geometry [4, 5]. Recently there have been considerable efforts [3, 6–20] to construct gravity from NC field theories. The emergent gravity has also been suggested to resolve the cosmological constant problem and dark energy [15, 21].

In section 3, we put the arguments in section 2 on a firm foundation using the background independent formulation of NC gauge theory [22, 23]. In section 3.1, we first clarify based on the argument in [14] that the emergent gravity from NC gauge theory is essentially a large N duality consistent with the AdS/CFT duality [24]. And then we move onto the geometric representation of NC field theory using the inner automorphism (1.4) of the NC spacetime (1.3). In section 3.2, we show how to explicitly determine a gravitational metric emerging from NC gauge fields and show that the equations of motion for NC gauge fields are mapped to the Einstein equations for the emergent metric. This part consists of our main new results generalizing the emergent gravity in [3, 12] for self-dual gauge fields. In the course of the derivation, we find that NC gauge fields induce an exotic form

JHEP05(2009)012

of energy, dubbed as the Liouville energy-momentum tensor. A simple analysis shows that this Liouville energy mimics the several aspects of dark energy, so we suggest the energy as a plausible candidate of dark energy. In section 3.3, the emergent gravity is further generalized to the nontrivial background of nonconstant θ induced by an inhomogenous condensation of gauge fields. In section 3.4, we discuss the spacetime picture revealed from NC gauge fields. We also confirm the observation in [15] that the emergent gravity reveals a remarkably beautiful and consistent picture about the dynamical origin of flat spacetime.

In section 4 we speculate how to understand "Time" and matter fields in the context of emergent geometry. As a first step, we elucidate in section 4.1 how the well-known 'minimal coupling' of matters with gauge fields can be understood as a symplectic geometry in phase space. There are two important works [25, 26] for this understanding. Based on the symplectic geometry of particles, in section 4.2, we suggest a K-theory picture for matter fields such as quarks and leptons adopting the Fermi-surface scenario in [27, 28] where non-Abelian gauge fields are understood as collective modes acting on the matter fields.

In section 5, we address the problem on the existence of spin-2 bound states which presupposes the basis of emergent gravity. Although we don't know any rigorous proof, we outline some positive evidences for the bound states using the relation to the AdS/CFT duality. We further notice an interesting similarity between the BCS superconductivity [29] and the emergent gravity about some dynamical mechanism for the spin-0 and spin-2 bound states, respectively. See the table 3. We also discuss the issues on the Lorentz symmetry breaking and the nonlocality in NC field theories from the viewpoint of emergent spacetime.

In section 6, we summarize the message uncovered by the emergent gravity picture with some closing remarks.

The calculational details in section 3 are deferred to two appendices. In appendix A we give a self-contained proof of the equivalence between self-dual NC electromagnetism and self-dual Einstein gravity, first shown in [12], for completeness. The self-dual case will provide a clear picture to appreciate what the emergent gravity is, which will also be useful to consider a general situation of emergent gravity. In appendix B the equivalence is generalized to arbitrary NC gauge fields.

2 Geometrization of forces

One of the guiding principles in modern physics is the geometrization of forces, i.e., to view physical forces as a reflection of the curvature of the geometry of spacetime or internal space. In this line of thought, gravity is quite different from the other three forces - the electromagnetic, the weak, and the strong interactions. It is a manifestation of the curvature of spacetime while the other three are a manifestation of the curvature of internal spaces. If it makes sense to pursue a unification of forces, in which the four forces are different manifestations of a single force, it would be desirable to reconcile gravity with the others and to find a general categorical structure of physical forces: Either to find a rationale that gravity is not a fundamental force or to find a framework that the other three forces are also geometrical properties of spacetime. We will show these two features are simultaneously realized in NC spacetime, at least, for the electromagnetism.

2.1 Einstein's happiest thought

The geometrization of forces is largely originated with Albert Einstein, whose general theory of relativity is to view the gravity as a metric field of spacetime which is determined by the distribution of matter and energy. The remarkable vision of gravity in terms of the geometry of spacetime has been based on the local equivalence of gravitation and inertia, or the local cancellation of the gravitational field by local inertial frames - the equivalence principle. Einstein once recalled that the equivalence principle was the happiest thought of his life.

The equivalence principle guarantees that it is "always" possible at any spacetime point of interest to find a coordinate system, say ξ^α, such that the effects of gravity will disappear over a differential region in the neighborhood of that point. (Precisely speaking, the neighborhood should be taken small enough so that the variation of gravity within the region may be neglected.) For a particle moving freely under the influence of purely gravitational force, the equation of motion in terms of the freely falling coordinate system ξ^α is thus

$$\frac{d^2\xi^\alpha}{d\tau^2} = 0 \tag{2.1}$$

with $d\tau$ the proper time

$$d\tau^2 = \eta_{\alpha\beta} d\xi^\alpha d\xi^\beta. \tag{2.2}$$

We will use the metric $\eta_{\alpha\beta}$ with signature $(- + + \cdots)$ throughout the paper.

Suppose that we perform a coordinate transformation to find the corresponding equations in a laboratory at rest, which may be described by a Cartesian coordinate system x^μ. The freely falling coordinates ξ^α are then functions of the x^μ, that is, $\xi^\alpha = \xi^\alpha(x)$. The freely falling particle in the laboratory coordinate system now obeys the equation of motion

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0 \tag{2.3}$$

where

$$d\tau^2 = g_{\mu\nu}(x) dx^\mu dx^\nu \tag{2.4}$$

and

$$g_{\mu\nu}(x) = \eta_{\alpha\beta}\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial\xi^\beta}{\partial x^\nu}. \tag{2.5}$$

It turns out that eq. (2.3) is the geodesic equation moving on the shortest possible path between two points through the curved spacetime described by the metric (2.5). In the end the gravitational force manifests itself only as the geometry of spacetime.

In accordance with the principle of general covariance the laws of physics must be independent of the choice of spacetime coordinates. That is, eq. (2.3) is true in all coordinate systems. For example, under a coordinate transformation $x^\mu \to x'^\mu$, the metric transforms into

$$g'_{\mu\nu}(x') = \frac{\partial x^\lambda}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} g_{\lambda\sigma}(x) \tag{2.6}$$

and eq. (2.3) transforms into the geodesic equation in the spacetime described by the metric (2.6). The significance of the equivalence principle in conjunction with the principle of covariance lies in its statement that there "always" exists a locally inertial frame at an arbitrary point P in spacetime where $g'_{\alpha\beta}(P) = \eta_{\alpha\beta}$ and $\Gamma'^{\mu}_{\alpha\beta}(P) = 0$. But the second derivatives of $g'_{\alpha\beta}$ at P cannot all be set to zero unless the spacetime is flat. This coordinate system is precisely the freely falling coordinates ξ^α in eq. (2.1), i.e., $\xi^\alpha = x'^\alpha(x)$, so the metric at P in the original system can consistently be written as the form (2.5).

But a routine calculation using the metric (2.5) leads to identically vanishing curvature tensors. Thus one may claim that the geometry described by the metric (2.5) is always flat. Of course it should not be the case. Remember that the metric (2.5) in the x-coordinate system should be understood at a point P since it has been obtained from the local inertial frame ξ^α where $g'_{\alpha\beta}(P) = \eta_{\alpha\beta}$ and $\Gamma'^{\mu}_{\alpha\beta}(P) = 0$ are satisfied only at that point. In order to calculate the curvature tensors correctly, one needs to extend the local inertial frame at P to an infinitesimal neighborhood. A special and useful realization of such a local inertial frame is a Riemann normal coordinate system [30] (where we choose the point P as a coordinate origin, i.e., $\xi^\alpha|_P = x^\mu|_P = 0$)

$$\xi^\alpha(x) = x^\alpha + \frac{1}{2}\Gamma^\alpha_{\mu\nu}(P)x^\mu x^\nu + \frac{1}{6}\big(\Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\lambda} + \partial_\lambda \Gamma^\alpha_{\mu\nu}\big)(P)x^\mu x^\nu x^\lambda + \cdots\,, \tag{2.7}$$

which can be checked using eq. (2.6) with the identification $x'^\alpha = \xi^\alpha$. One can then arrive at a metric

$$g'_{\alpha\beta}(x) = \eta_{\alpha\beta} - \frac{1}{3}R_{\alpha\mu\beta\nu}(P)x^\mu x^\nu - \frac{1}{6}D_\lambda R_{\alpha\mu\beta\nu}(P)x^\lambda x^\mu x^\nu + \cdots\,. \tag{2.8}$$

2.2 Darboux theorem as the equivalence principle in symplectic geometry

What about other forces ? Is it possible to realize, for example, the electromagnetism as a geometrical property of spacetime like gravity ? To be specific, we are wondering whether or not there "always" exists any coordinate transformation to eliminate the electromagnetic force at least locally. The usual wisdom says no since there is no analogue of the equivalence principle for the geometrization of the electromagnetic force. But one has to recall that this wisdom has been based on the usual concept of geometry, i.e., Riemannian geometry in commutative spacetime. Surprisingly, the conventional wisdom turns out to be no longer true in NC spacetime, which is based on symplectic geometry in sharp contrast to the Riemannian geometry.

We will show that it is "always" possible to find a coordinate transformation to eliminate locally the electromagnetic force if and only if spacetime supports a symplectic structure, viz., NC spacetime. To be definite, we will proceed with string theory although an elegant and rigorous approach can be done using the formalism of deformation quantization [31]. See [3] for some arguments based on the latter approach.

A scheme to introduce gauge fields in string theory is by means of boundary interactions or via boundary conditions of open strings, aside from through the Kaluza-Klein compactifications in type II or heterotic string theories. With a compact notation, the open or closed string action reads as[1]

$$S = \frac{1}{4\pi\alpha'} \int_\Sigma |dX|^2 - \int_\Sigma B - \int_{\partial\Sigma} A, \tag{2.9}$$

where $X : \Sigma \to M$ is a map from an open or closed string worldsheet Σ to a target spacetime M and $B(\Sigma) = X^* B(M)$ and $A(\partial\Sigma) = X^* A(M)$ are pull-backs of spacetime fields to the worldsheet Σ and the worldsheet boundary $\partial\Sigma$, respectively.

The string action (2.9) respects the following local symmetries.

(I) Diff(M)-symmetry:

$$X \to X' = X'(X) \in \mathrm{Diff}(M) \tag{2.10}$$

and the corresponding transformations of target fields B and A including also a target metric (hidden) in the first term of eq. (2.9).

(II) Λ-symmetry:

$$(B,\ A) \to (B - d\Lambda,\ A + \Lambda) \tag{2.11}$$

where the gauge parameter Λ is a one-form in M. A simple application of Stokes' theorem immediately verifies the symmetry (2.11). Note that the Λ-symmetry is present only when $B \neq 0$. When $B = 0$, the symmetry (2.11) is reduced to $A \to A + d\lambda$, which is the ordinary U(1) gauge symmetry.

The above two local symmetries in string theory must also be realized as the symmetries in low energy effective theory. We well understand the root of the symmetry (2.10) since the string action (2.9) describes a gravitational theory in target spacetime. The diffeomorphism symmetry (2.10) certainly signifies the emergence of gravity in the target space M. A natural question is then what is a root of the Λ-symmetry (2.11).

Unfortunately, as far as we know, there has been no serious investigation about a physical consequence of the symmetry (2.11). As a provoking comment, let us first point out that the Λ-symmetry (2.11) is as large as the Diff(M)-symmetry (2.10) (supposing that the rank of B is equal to the dimension of M) and is present only when $B \neq 0$, so a stringy symmetry by nature. Indeed this is a broad hint that there will be a radical change of physics when $B \neq 0$ — the new physics due to the θ-deformation in the table 1.

To proceed with a general context, let us first discuss a geometrical interpretation of the Λ-symmetry without specifying low energy effective theories. Suppose that the two-form $B \in \Lambda^2(M)$ is closed in M, i.e., $dB = 0$, and nondegenerate, i.e., nowhere vanishing in M.[2] One can then regard the two-form B as a symplectic structure on M and the pair (M, B) as a symplectic manifold. The symplectic geometry is a less intuitive type of

[1]Although we will focus on the open string theory, our arguments in this section also hold for a closed string theory where the string worldsheet Σ is a compact Riemann surface without boundary, so the last term in eq. (2.9) is absent.

[2]In string theory, $H = dB \in \Lambda^3(M)$ is not necessarily zero. We don't know much about this case, so we will restrict to the symplectic case. But the connection with the generalized geometry, to be shortly discussed later, might be helpful to understand more general cases.

JHEP05(2009)012

geometry but it should be familiar with classical mechanics, especially, the Hamiltonian mechanics [32] and, more prominently, quantum mechanics.

The symplectic geometry respects an important property, known as the Darboux theorem [33], stating that every symplectic manifold of the same dimension is locally indistinguishable. More precisely, let (M, ω) be a symplectic manifold. Then in a neighborhood of each $P \in M$, there is a local coordinate chart in which ω is a constant, i.e., $(M, \omega) \cong (\mathbf{R}^{2n}, \sum dq^i \wedge dp_i)$. For our purpose, we will use its refined version - the Moser lemma [34] - describing a cohomological condition for two symplectic structures to be equivalent. Given two-forms ω and ω' such that $[\omega] = [\omega'] \in H^2(M)$ and $\omega_t = \omega + t(\omega' - \omega)$ is symplectic $\forall t \in [0,1]$, then there exists a diffeomorphism $\phi_t : M \to M$ such that $\phi_t^*(\omega_t) = \omega$. This implies that all ω_t are related by coordinate transformations generated by a vector field X_t satisfying

$$\iota_{X_t}\omega_t + A = 0 \tag{2.12}$$

where $\omega' - \omega = dA$. In terms of local coordinates, there always exists a coordinate transformation ϕ_1 whose pullback maps $\omega' = \omega + dA$ to ω, i.e., $\phi_1 : y \mapsto x = x(y)$ so that

$$\frac{\partial x^\alpha}{\partial y^a}\frac{\partial x^\beta}{\partial y^b}\omega'_{\alpha\beta}(x) = \omega_{ab}(y). \tag{2.13}$$

The Moser lemma (2.13) stating that the symplectic manifolds (M, ω_0) and (M, ω_1) are strongly isotopic is a global statement and will be applied to our problem as follows. For a symplectic manifold $(M, \omega_1 = B + F)$ where $F = dA$, by the Darboux theorem, one can always find a local coordinate chart $(U; y^1, \cdots, y^{2n})$ centered at $p \in M$ and valid on the neighborhood U such that $\omega_0(p) = \frac{1}{2}B_{ab}dy^a \wedge dy^b$ where B_{ab} is a constant symplectic matrix of rank $2n$. Then there are two symplectic structures on U; the given $\omega_1 = B + F$ and $\omega_0 = B$. Consider a smooth family $\omega_t = \omega_0 + t(\omega_1 - \omega_0)$ of symplectic forms joining ω_0 to ω_1. Now the Moser lemma (2.13) implies that there exists a global diffeomorphism $\phi : M \times \mathbf{R} \to M$ such that $\phi_t^*(\omega_t) = \omega_0,\ 0 \le t \le 1$. If there exists such a diffeomorphism, in terms of the associated time-dependent vector field $X_t \equiv \frac{d\phi_t}{dt} \circ \phi_t^{-1}$, one would have for all $0 \le t \le 1$ that $\mathcal{L}_{X_t}\omega_t + \frac{d\omega_t}{dt} = 0$ which can be reduced to eq. (2.12). One can pointwise solve the Moser's equation (2.12) to obtain a unique smooth family of vector fields $X_t,\ 0 \le t \le 1$, generating the global diffeomorphism ϕ_t satisfying $\frac{d\phi_t}{dt} = X_t \circ \phi_t$. So everything boils down to solving the Moser's equation (2.12) for X_t.

First one may solve the equation (2.12) at $t \to 0$ to determine $X_0 = X_0(y)$ on U in terms of the Darboux coordinates y^a and extend to all $0 \le t \le 1$ by integration [35]. After integration, one can find a local isotopy $\phi : U \times [0,1] \to M$ with $\phi_t^*(\omega_t) = \omega_0$ for all $t \in [0,1]$. Let us denote the resulting coordinate transformation $\phi_1(y)$ on U generated by the vector field X_1 as $x^a(y) = y^a + X_1^a(y)$. (Compare the result with eq. (2.22) where $X_1^a(y) := \theta^{ab}\widehat{A}_b(y)$.) This is the result we want to get from the data $(M, \omega_1 = B + F)$ by performing a coordinate transformation (2.13) onto a local Darboux chart. Therefore sometimes we will simply refer the Darboux theorem to eq. (2.13) in a loose sense as long as the physical meaning is clear.

The string action (2.9) indicates that, when $B \neq 0$, its natural group of symmetries includes not only the diffeomorphism (2.10) in Riemannian geometry but also the

(I) Riemannian geometry	(II) Symplectic geometry
Riemannian manifold (M, g): M a smooth manifold and $g : TM \otimes TM \to \mathbf{R}$ a nondegenerate symmetric bilinear form	Symplectic manifold (M, ω): M a smooth manifold and $\omega \in \Lambda^2(M)$ a nondegenerate closed 2-form, i.e., $d\omega = 0$
Equivalence principle: Locally, $(M, g) \cong (\mathbf{R}^{2n}, \sum dx^\mu \otimes dx_\mu)$	Darboux theorem: Locally, $(M, \omega) \cong (\mathbf{R}^{2n}, \sum dq^i \wedge dp_i)$

Table 2. Riemannian geometry vs. Symplectic geometry.

Λ-symmetry (2.11) in symplectic geometry. According to the Darboux theorem (precisely, the Moser lemma stated above), the local change of symplectic structure due to the Λ-symmetry (2.11) (or the B-field transformation) can always be translated into a diffeomorphism symmetry as in eq. (2.13). This fact implies that the Λ-symmetry (2.11) should be considered as a par with diffeomorphisms. It turns out [3] that the Darboux theorem in symplectic geometry plays the same role as the equivalence principle in general relativity for the geometrization of the electromagnetic force. These geometrical structures inherent in the string action (2.9) are summarized below.

Therefore we need a generalized geometry when $B \neq 0$ which treats both Riemannian geometry and symplectic geometry on equal footing.[3] Such kind of generalized geometry was introduced by N. Hitchin [4] in 2002 and further developed by M. Gualtieri and G. R. Cavalcanti [5]. Generalized complex geometry unites complex and symplectic geometries such that it interpolates between a complex structure J and a symplectic structure ω by viewing each as a complex (or symplectic) structure $\mathcal{J}$ on the direct sum of the tangent and cotangent bundle $E = TM \oplus T^*M$. A generalized complex structure $\mathcal{J} : E \to E$ is a generalized almost complex structure, satisfying $\mathcal{J}^2 = -1$ and $\mathcal{J}^* = -\mathcal{J}$, whose sections are closed under the Courant bracket[4]

$$[X + \xi, Y + \eta]_C = [X, Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}d\big(\iota_X\eta - \iota_Y\xi\big), \tag{2.14}$$

where $\mathcal{L}_X$ is the Lie derivative along the vector field X and $d\,(\iota)$ is the exterior (interior) product.

[3]A Riemannian geometry is defined by a pair (M, g) where the metric g encodes all geometric informations while a symplectic geometry is defined by a pair (M, ω) where the 2-form ω encodes all. See the table 2. A basic concept in Riemannian geometry is a distance defined by the metric. One may identify this distance with a geodesic worldline of a "particle" moving in M. On the contrary, a basic concept in symplectic geometry is an area defined by the symplectic structure. One may regard this area as a minimal worldsheet swept by a "string" moving in M. Amusingly, the Riemannian geometry is probed by particles while the symplectic geometry would be probed by strings. But we know that a Riemannian geometry (or gravity) is emergent from strings ! This argument, though naive, glimpses the reason why the θ-deformation in the table 1 goes parallel to the α'-deformation.

[4]When $H = dB$ is not zero, the Courant bracket on E is 'twisted' by the real, closed 3-form H in the following way

$$[X + \xi, Y + \eta]_H = [X + \xi, Y + \eta]_C + \iota_Y\iota_X H.$$

See [5] for more details, in particular, a relation to gerbes.

An important point in generalized geometry is that the symmetries of E, i.e., the endomorphisms of E (the group of orthogonal Courant automorphisms of E), are the composition of a diffeomorphism of M and a B-field transformation defined by $e^B(X+\xi) = X+\xi+\iota_X B$ for any $X+\xi \in E$, where B is an arbitrary closed 2-form. This B-field transformation can be identified with the Λ-symmetry (2.11) as follows. Let (M,B) be a symplectic manifold where $B = d\xi$, locally, by the Poincaré lemma. The Λ-symmetry (2.11) can then be understood as a shift of the canonical 1-form, $\xi \to \xi - \Lambda$, which is the B-field transformation with the identification $\Lambda = -\iota_X B$. With this notation, the B-field transformation is equivalent to $B \to B + \mathcal{L}_X B$ since $dB = 0$. We thus see that the generalized complex geometry provides a natural geometric framework to incorporate simultaneously the two local symmetries in eq. (2.10) and eq. (2.11). That is,

$$\text{Courant automorphism} = \text{Diff(M)} \oplus \Lambda - \text{symmetry}. \tag{2.15}$$

One can introduce a generalized metric on $TM \oplus T^*M$ by reducing the structure group $U(n,n)$ to $U(n) \times U(n)$. It turns out [5] that the metric on $TM \oplus T^*M$ compatible with the natural pairing $\langle X+\xi, Y+\eta\rangle = \frac{1}{2}\big(\xi(Y)+\eta(X)\big)$ is equivalent to a choice of metric g on TM and 2-form B.[5] We now introduce a DBI "metric" $g+\kappa B : TM \to T^*M$ which maps X to $\xi = (g+\kappa B)(X)$. Consider the Courant automorphism (2.15) which is a combination of a B-field transformation followed by a diffeomorphism $\phi : M \to M$

$$X + \xi \to \phi_*^{-1} X + \phi^*(\xi + \iota_X B). \tag{2.16}$$

The above action transforms the DBI metric $g + \kappa B$ according to

$$g + \kappa B \to \phi^*\Big(g + \kappa(B + \mathcal{L}_X B)\Big). \tag{2.17}$$

The Moser lemma (2.13) then implies that there always exists a diffeomorphism ϕ such that $\phi^*(B + \mathcal{L}_X B) = B$. In terms of local coordinates $\phi : y \to x = x(y)$, eq. (2.17) then reads as

$$(g + \kappa B')_{\alpha\beta}(x) = \frac{\partial y^a}{\partial x^\alpha}\Big(g'_{ab}(y) + \kappa B_{ab}(y)\Big)\frac{\partial y^b}{\partial x^\beta} \tag{2.18}$$

where $B' = B + \mathcal{L}_X B$ and

$$g'_{ab}(y) = \frac{\partial x^\alpha}{\partial y^a}\frac{\partial x^\beta}{\partial y^b} g_{\alpha\beta}(x). \tag{2.19}$$

One can immediately see that the diffeomorphism (2.18) between two different DBI metrics is a direct result of the Moser lemma (2.13). We will see that the identity (2.18) leads to a remarkable relation between symplectic (or Poisson) geometry and complex (or Riemannian) geometry.

[5] A reduction to $U(n) \times U(n)$ is equivalent to the existence of two generalized almost complex structures $\mathcal{J}_1, \mathcal{J}_2$ where $\mathcal{J}_1$ and $\mathcal{J}_2$ commute and a generalized Kähler metric $G = -\mathcal{J}_1\mathcal{J}_2$ is positive definite. This structure is known as the generalized Kähler or bi-Hermitian structure [5]. Any generalized Kähler metric G takes the form

$$G = \begin{pmatrix} -g^{-1}B & g^{-1} \\ g - Bg^{-1}B & Bg^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}\begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix},$$

which is the B-field transformation of a bare Riemannian metric g as long as the 2-form B is closed. Interestingly the metric part $g - Bg^{-1}B : TM \to T^*M$ in the generalized Kähler metric G is exactly of the same form as the open string metric in a B-field [22].

2.3 DBI action as a generalized geometry

We observed that the presence of a nowhere vanishing (closed) 2-form B in spacetime M calls for a generalized geometry, where the two local symmetries in eq. (2.15) are treated on equal footing. A crucial point in the generalized geometry is that the space $\Lambda^2(M)$ of closed 2-forms in M appears as a part of spacetime geometry, as embodied in eq. (2.18), in addition to the Diff(M) symmetry being a local isometry of Riemannian geometry. This suggests that, when $B \neq 0$, it is possible to realize a completely new geometrization of a physical force based on symplectic geometry rather than Riemannian geometry. So a natural question is: What is the force ?

We will show that the force is indeed the electromagnetic force and there exists a novel form of the equivalence principle, i.e., the Darboux theorem, for the geometrization of the electromagnetism. In other words, eq. (2.13) implies that there always exists a coordinate transformation to locally eliminate the electromagnetic force as long as the D-brane worldvolume M supports a symplectic structure B, i.e., M becomes a NC space. Furthermore, U(1) gauge transformations in NC spacetime become a 'spacetime' symmetry rather than an 'internal' symmetry, which already suggests that the electromagnetism in NC spacetime can be realized as a geometrical property of spacetime like gravity.

Let us now discuss the physical consequences of the generalized geometry, especially, the implications of the Λ-symmetry (2.11) in the context of the low energy effective theory of open strings in the background of an NS-NS 2-form B. We will use the effective field theory description in order to broadly illuminate what kind of new physics arises from a field theory in the B-field background, i.e., a NC field theory. It will provide a clear-cut picture about the new physics though it is not quite rigorous. In the next section we will put the arguments here on a firm foundation using the background independent formulation of NC gauge theory.

A low energy effective field theory deduced from the open string action (2.9) describes an open string dynamics on a $(p+1)$-dimensional D-brane worldvolume. The dynamics of D-branes is described by open string field theory whose low energy effective action is obtained by integrating out all the massive modes, keeping only massless fields which are slowly varying at the string scale $\kappa \equiv 2\pi\alpha'$. For a Dp-brane in closed string background fields, the action describing the resulting low energy dynamics is given by

$$S = \frac{2\pi}{g_s(2\pi\kappa)^{\frac{p+1}{2}}} \int d^{p+1}x \sqrt{\det(g+\kappa(B+F))} + \mathcal{O}(\sqrt{\kappa}\partial F, \cdots), \tag{2.20}$$

where $F = dA$ is the field strength of U(1) gauge fields. The DBI action (2.20) respects the two local symmetries, (2.10) and (2.11), as expected.

(I) Diff(M)-symmetry: Under a local coordinate transformation $\phi^{-1} : x^\alpha \mapsto x'^\alpha$ where worldvolume fields also transform in usual way

$$(B' + F')_{ab}(x') = \frac{\partial x^\alpha}{\partial x'^a}\frac{\partial x^\beta}{\partial x'^b}(B+F)_{\alpha\beta}(x) \tag{2.21}$$

together with the metric transformation (2.6), the action (2.20) is invariant.

(II) Λ-symmetry: One can easily see that the action (2.20) is invariant under the transformation (2.11) with any 1-form Λ.

Note that ordinary U(1) gauge symmetry is a special case of eq. (2.11) where the gauge parameter Λ is exact, namely, $\Lambda = d\lambda$, so that $B \to B$, $A \to A + d\lambda$. Indeed the U(1) gauge symmetry is a diffeomorphism (known as a symplectomorphism) generated by a vector field X satisfying $\mathcal{L}_X B = 0$. We see here that the gauge symmetry becomes a 'spacetime' symmetry rather than an 'internal' symmetry, as well as an infinite-dimensional and non-Abelian symmetry when B is nowhere vanishing. This fact unveils a connection between NC gauge fields and spacetime geometry.

The geometrical data of D-branes, that is a derived category in mathematics, are specified by the triple (M, g, B) where M is a smooth manifold equipped with a Riemannian metric g and a symplectic structure B. One can see from the action (2.20) that the data come only into the combination $(M, g, B) = (M, g + \kappa B)$, which is the DBI metric (2.17) to embody a generalized geometry. In fact the 'D-manifold' defined by the triple (M, g, B) describes the generalized geometry [4, 5] which continuously interpolates between a symplectic geometry ($|\kappa B g^{-1}| \gg 1$) and a Riemannian geometry ($|\kappa B g^{-1}| \ll 1$). An important point is that the electromagnetic force F should appear in the gauge invariant combination $\Omega = B + F$ due to the Λ-symmetry (2.11), as shown in eq. (2.20). Then the Darboux theorem (2.13) with the identification $\omega' = \Omega$ and $\omega = B$ states that one can "always" eliminate the electromagnetic force F by a suitable local coordinate transformation as far as the 2-form B is nondegenerate. Therefore the Darboux theorem in symplectic goemetry bears an analogy with the equivalence principle in section 2.1.

Let us represent the local coordinate transform $\phi : y \mapsto x = x(y)$ in eq. (2.13) as follows

$$x^a(y) \equiv y^a + \theta^{ab}\widehat{A}_b(y), \tag{2.22}$$

where θ^{ab} is a Poisson structure on M, i.e., $\theta^{ab} = \left(\frac{1}{B}\right)^{ab}$.[6] This particular form of expression has been motivated by the fact that $\omega'_{ab}(x) = \omega_{ab}(y)$ in the case of $F = dA = 0$, so the second term in eq. (2.22) should take care of the deformation of the symplectic structure coming from $F = dA$. As was shown above, U(1) gauge transformations are generated by a Hamiltonian vector field X_λ satisfying $\iota_{X_\lambda} B + d\lambda = 0$ and the action of X_λ on $x^a(y)$ is given by

$$\begin{aligned} \delta x^a(y) &\equiv X_\lambda(x^a) = \{x^a, \lambda\}_\theta \\ &= \theta^{ab}\big(\partial_b \lambda + \{\widehat{A}_b, \lambda\}_\theta\big), \end{aligned} \tag{2.23}$$

where the last expression presumes a constant θ^{ab}. The above transformation will be identified with the NC U(1) gauge transformation after a NC deformation, so $\widehat{A}_a(y)$ turns out to be NC gauge fields. The coordinates $x^a(y)$ in (2.22) will play a special role, since they are background independent [23] as well as gauge covariant [36].

[6]A Poisson structure is a skew-symmetric, contravariant 2-tensor $\theta = \theta^{ab}\partial_a \wedge \partial_b \in \bigwedge^2 TM$ which defines a skew-symmetric bilinear map $\{f, g\}_\theta = \langle \theta, df \otimes dg \rangle = \theta^{ab}\partial_a f \partial_b g$ for $f, g \in C^\infty(M)$, so-called, a Poisson bracket. So we get $\theta^{ab}(y) = \{y^a, y^b\}_\theta$.

We showed before that the local equivalence (2.13) between symplectic structures brings in the diffeomorphic equivalence (2.18) between two different DBI metrics, which in turn leads to a remarkable identity between DBI actions [37]:

$$\int d^{p+1}x\sqrt{\det\big(g(x)+\kappa(B+F)(x)\big)}=\int d^{p+1}y\sqrt{\det\big(h(y)+\kappa B(y)\big)}. \tag{2.24}$$

Note that gauge field fluctuations now appear as an induced metric on the brane given by

$$h_{ab}(y)=\frac{\partial x^\alpha}{\partial y^a}\frac{\partial x^\beta}{\partial y^b}g_{\alpha\beta}(x). \tag{2.25}$$

The identity (2.24) can also be obtained by considering the coordinate transformations (2.6) and (2.21) satisfying $(B'+F')_{ab}(x')=B_{ab}(x')$. This kind of coordinate transformation always exists thanks to the Darboux theorem (2.13). Note that all these underlying structures are very parallel to general relativity (see section 2.1). For instance, considering the fact that a diffeomorphism $\phi\in\mathrm{Diff(M)}$ acts on E as $X+\xi\mapsto\phi_*^{-1}X+\phi^*\xi$, we see that the covariant coordinates $x^a(y)$ in eq. (2.22) correspond to the locally inertial coordinates $\xi^\alpha(x)$ in eq. (2.1) while the coordinates y^a play the same role as the laboratory Cartesian coordinates x^μ in eq. (2.3).

We will now discuss important physical consequences we can get from the identity (2.24).

(1) The identity (2.24) says that gauge field fluctuations on a rigid D-brane are equivalent to dynamical fluctuations of the D-brane itself without gauge fields. Indeed this picture is omnipresent in string theory with the name of open-closed string duality although it is not formulated in this way.

(2) The identity (2.24) cannot be true when $B=0$, i.e., spacetime is commutative. In this case the Λ-symmetry is reduced to ordinary U(1) gauge symmetry. The gauge symmetry has no relation to a diffeomorphism symmetry and it is just an internal symmetry rather than a spacetime symmetry.

(3) Let us consider a curved D-brane in a constant B-field background whose shape is described by an induced metric h_{ab}. We may consider the right-hand side of eq. (2.24) with a constant $B_{\rm const}$ as the corresponding DBI action. The induced metric h_{ab} can be represented as in eq. (2.25) with a flat metric $g_{\alpha\beta}(x)=\delta_{\alpha\beta}$. The nontrivial shape of the curved D-brane described by the metric h_{ab} can then be translated in the left-hand side of eq. (2.24) into a nontrivial condensate of gauge fields on a flat D-brane given by

$$B_{ab}(x)=\big(B_{\rm const}+F_{\rm back}(x)\big)_{ab}. \tag{2.26}$$

The converse is also suggestive. Any symplectic 2-form on a noncompact space can be written as the form (2.26) where $B_{\rm const}$ is an asymptotic value of the 2-form $B_{ab}(x)$, i.e., $F_{\rm back}(x)\to 0$ at $|x|\to\infty$. And the gauge field configuration $F_{\rm back}(x)$ can be interpreted as a curved D-brane manifold in the $B_{\rm const}$ background. Thus

JHEP05(2009)012

we get an intriguing result that a curved D-brane with a canonical symplectic 2-form (or a constant Poisson structure) is equivalently represented as a flat D-brane with an inhomogeneous symplectic 2-form (or a nonconstant Poisson structure). Our argument here also implies a fascinating result that $B_{\rm const}$, a uniform condensation of gauge fields in a vacuum, would be a 'source' of flat spacetime. Later we will return to this point with an elaborated viewpoint.

(4) One can expand the right-hand side of eq. (2.24) around the background B, arriving at the following result [37]

$$\begin{aligned}&\int d^{p+1}y\sqrt{\det\big(h(y)+\kappa B(y)\big)}\\&\quad=\int d^{p+1}y\sqrt{\det\big(\kappa B\big)}\Big(1+\frac{1}{4\kappa^2}g_{ac}g_{bd}\{x^a,x^b\}_\theta\{x^c,x^d\}_\theta+\cdots\Big)\end{aligned}\tag{2.27}$$

where $\{x^a,x^b\}_\theta$ is a Poisson bracket (defined in footnote 6) between the covariant coordinates (2.22). For constant B and g, eq. (2.27) is equivalent to the IKKT matrix model [38] after a quantization *à la* Dirac, i.e., $\{x^a,x^b\}_\theta\Rightarrow -i[\widehat{x}^a,\widehat{x}^b]_\star$, which is believed to describe the nonperturbative dynamics of the type IIB string theory. Furthermore one can show that eq. (2.27) reduces to a NC gauge theory, using the relation

$$[\widehat{x}^a,\widehat{x}^b]_\star=-i\Big(\theta(\widehat{F}-B)\theta\Big)^{ab}\tag{2.28}$$

where the NC field strength is given by

$$\widehat{F}_{ab}=\partial_a\widehat{A}_b-\partial_b\widehat{A}_a-i[\widehat{A}_a,\widehat{A}_b]_\star.\tag{2.29}$$

Therefore the identity (2.24) is, in fact, the Seiberg-Witten equivalence between commutative and NC DBI actions [22].

(5) It was explicitly demonstrated in [3, 12] how NC gauge fields manifest themselves as a spacetime geometry, as eq. (2.27) glimpses this geometrization of the electromagnetic force. Surprisingly it turns out [12] that self-dual electromagnetism in NC spacetime is equivalent to self-dual Einstein gravity. (We rigorously show this equivalence in appendix A.) For example, U(1) instantons in NC spacetime are actually gravitational instantons [11]. This picture also reveals a beautiful geometrical structure that self-dual NC electromagnetism perfectly fits with the twistor space describing curved self-dual spacetime. The deformation of symplectic (or Kähler) structure of a self-dual spacetime due to the fluctuation of gauge fields appears as that of complex structure of the twistor space.

(6) All these properties appearing in the geometrization of electromagnetism may be summarized in the context of derived category. More closely, if M is a complex manifold whose complex structure is given by J, we see that dynamical fields in the left-hand side of eq. (2.24) act only as the deformation of symplectic structure $\Omega(x)=B+F(x)$ in the triple (M,J,Ω), while those in the right-hand side of eq. (2.24)

appear only as the deformation of complex structure $J'(y)$ in the triple (M', J', B) through the metric (2.25). In this notation, the identity (2.24) can thus be written as follows

$$(M, J, \Omega) \cong (M', J', B). \tag{2.30}$$

The equivalence (2.30) is very reminiscent of the homological mirror symmetry [39], stating the equivalence between the category of A-branes (derived Fukaya category corresponding to the triple (M, J, Ω)) and the category of B-branes (derived category of coherent sheaves corresponding to the triple (M', J', B)).

There is a subtle but important difference between the Riemannian geometry and symplectic geometry. Strictly speaking, the equivalence principle in general relativity is a point-wise statement at any given point P while the Darboux theorem in symplectic geometry is defined in an entire neighborhood around P. This is the reason why there exist local invariants, e.g., curvature tensors, in Riemannian geometry while there is no such kind of local invariant in symplectic geometry.[7] This raises a keen puzzle about how Riemannian geometry is emergent from symplectic geometry though their local geometries are in sharp contrast to each other.

We suggest a following resolution. A symplectic structure B is nowhere vanishing. In terms of physicist language, this means that there is an (inhomogeneous in general) condensation of gauge fields in a vacuum, i.e.,

$$\langle B_{ab}(x) \rangle_{\rm vac} = \theta^{-1}_{ab}(x). \tag{2.31}$$

Let us consider a constant symplectic structure for simplicity. The background (2.31) then corresponds to a uniform condensation of gauge fields in a vacuum given by $\langle A^0_a \rangle_{\rm vac} = -B_{ab} y^b$. It will be suggestive to rewrite the covariant coordinates (2.22) as (actually to invoke a renowned Goldstone boson $\varphi = \langle \varphi \rangle + h$)[8]

$$x^a(y) = \theta^{ab} \Big(-\langle A^0_b \rangle_{\rm vac} + \widehat{A}_b(y) \Big). \tag{2.32}$$

This naturally suggests some sort of spontaneous symmetry breaking where y^a are vacuum expectation values of $x^a(y)$, specifying the background (2.31) as usual, and $\widehat{A}_b(y)$ are fluctuating (dynamical) coordinates (fields).

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[7] If the equivalence principle held over an entire neighborhood of a point P, curvature tensors would identically vanish. Indeed the existence of local invariants such as Riemann curvature tensors results from the implicit assumption that it is always possible to discriminate total gravitational fields between two arbitrary nearby spacetime points (see section 2.1). This exhibits a sign that there will be a serious conflict between the equivalence principle and the Heisenberg's uncertainty principle. In this perspective, it seems like a vain attempt to mix with water and oil to try to quantize Einstein gravity itself, which is based on Riemann curvature tensors of which the equivalence principle is in the heart.

[8] In this respect, it would be interesting to quote a recent comment of A. Zee [40]: "The basic equation for the graviton field has the same form $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. This naturally suggests that $\eta_{\mu\nu} = \langle g_{\mu\nu} \rangle$ and perhaps some sort of spontaneous symmetry breaking." We will show later that this pattern is not an accidental happening.

JHEP05(2009)012

Note that the vacuum (2.31) picks up a particular symplectic structure, introducing a typical length scale $||\theta|| = l_{nc}^2$. This means that the Λ-symmetry G in eq. (2.11) is spontaneously broken to the symplectomorphism H preserving the vacuum (2.31) [3]. The Λ-symmetry is the local equivalence between two symplectic structures belonging to the same cohomology class. But the transformation in eq. (2.11) will not preserve the vacuum (2.31) except its subgroup generated by the gauge parameter $\Lambda = d\lambda$ which is equal to the NC U(1) gauge symmetry (2.23).[9] So the deformations of the vacuum manifold (2.31) by NC gauge fields take values in the coset space G/H, which is equivalent to the gauge orbit space of NC gauge fields or the physical configuration space of NC electromagnetism [3]. The spontaneous symmetry breaking also explains why only ordinary U(1) gauge symmetry is observed at large scales $\gg l_{nc}$. We argued in [3] that the spontaneous symmetry breaking (2.31) will explain why Einstein gravity, carrying local curvature invariants, can emerge from symplectic geometry.[10] In other words, Riemannian geometry would simply be a result of coarse-graining of symplectic geometry at the scales $\gtrsim l_{nc}$ like as the Einstein gravity in string theory where the former simply corresponds to the limit $\alpha' \to 0$.

3 Emergent gravity

Sometimes a naive reasoning also suggests a road in mist. What is quantum gravity ? Quantum gravity means to quantize gravity. Gravity, according to Einstein's general relativity, is the dynamics of spacetime geometry which is usually described by a Hausdorff space M while quantization *à la* Dirac will require a phase space structure of spacetime as a prequantization. The phase space structure of spacetime M can be specified by introducing a symplectic structure ω on M. Therefore our naive reasoning implies that the pair (M, ω), a symplectic manifold, might be a proper starting point for quantum gravity, where fluctuations of spacetime geometry would be fluctuations of the symplectic structure ω and the quantization of symplectic manifold (M, ω) could be performed via the deformation quantization *à la* Kontsevich [31].[11] This state of art is precisely the situation we have encountered in the previous section for the generalized geometry emerging from the string theory (2.9) when $B \neq 0$.

A symplectic structure $B = \frac{1}{2} B_{ab} dy^a \wedge dy^b$ defines a Poisson structure $\theta^{ab} \equiv (B^{-1})^{ab}$ on M (see footnote 6) where $a, b = 1, \ldots, 2n$. From now on, we will refer to a constant symplectic structure unless otherwise specified. The Dirac quantization with respect to the Poisson structure θ^{ab} then leads to a quantum phase space (1.3). And the argument in

[9] We will show later that a constant shift of the symplectic structure, $B \to B' = B + \delta B$, does not affect any physics, so a symmetry of the theory, although it readjusts the vacuum (2.31).

[10] Here we are not saying that symplectic geometry is missing an important ingredient. Instead our physics simply requires to distinguish the background (nondynamical) and fluctuating (dynamical) parts of a symplectic structure. This will be a typical feature appearing in a background independent theory.

[11] This quantization scheme is different from the usual canonical quantization of gravity where metrics g and their conjugates π_g constitute fundamental variables for quantization, i.e., a phase space (g, π_g). We believe that the conventional quantization scheme is much like an escapade to quantize an elasticity of solid (e.g., sound waves) or hydrodynamics and it is supposed to be failed due to the choice of wrong variables for quantization, since it turns out that Riemannian metrics are not fundamental variables but collective (or composite) variables.

JHEP05(2009)012

section 2.3 also explains why a condensation of gauge fields in a vacuum, eq. (2.31), gives rise to the NC spacetime (1.3), i.e.,

$$\langle B_{ab}\rangle_{\rm vac} = (\theta^{-1})_{ab} \;\;\Leftrightarrow\;\; [y^a, y^b]_\star = i\theta^{ab} \;\;\Leftrightarrow\;\; [a_i, a_j^\dagger] = \delta_{ij}, \tag{3.1}$$

where a_i and $a_j^\dagger$ with $i, j = 1, \cdots, n$ are annihilation and creation operators, respectively, in the Heisenberg algebra of an n-dimensional harmonic oscillator.

It is a well-known fact from quantum mechanics that the representation space of NC $\mathbf{R}^{2n}$ is given by an infinite-dimensional, separable Hilbert space

$$\mathcal{H} = \{|\vec{n}\rangle \equiv |n_1, \cdots, n_n\rangle, \; n_i = 0, 1, \cdots\} \tag{3.2}$$

which is orthonormal, i.e., $\langle\vec{n}|\vec{m}\rangle = \delta_{\vec{n}\vec{m}}$ and complete, i.e., $\sum_{\vec{n}=0}^\infty |\vec{n}\rangle\langle\vec{n}| = 1$. Note that every NC space can be represented as a theory of operators in the Hilbert space $\mathcal{H}$, which consists of NC $\star$-algebra $\mathcal{A}_\theta$ like as a set of observables in quantum mechanics. Therefore any field $\widehat{\Phi} \in \mathcal{A}_\theta$ in the NC space (3.1) becomes an operator acting on $\mathcal{H}$ and can be expanded in terms of the complete operator basis

$$\mathcal{A}_\theta = \{|\vec{n}\rangle\langle\vec{m}|, \; n_i, m_j = 0, 1, \cdots\}, \tag{3.3}$$

that is,

$$\widehat{\Phi}(y) = \sum_{\vec{n},\vec{m}} \Phi_{\vec{n}\vec{m}} |\vec{n}\rangle\langle\vec{m}|. \tag{3.4}$$

One may use the 'Cantor diagonal method' to put the n-dimensional positive integer lattice in $\mathcal{H}$ into a one-to-one correspondence with the infinite set of natural numbers (i.e., 1-dimensional positive integer lattice): $|\vec{n}\rangle \leftrightarrow |n\rangle$, $n = 1, \cdots, N \to \infty$. In this one-dimensional basis, eq. (3.4) can be relabeled as the following form

$$\widehat{\Phi}(y) = \sum_{n,m=1}^\infty \Phi_{nm} \, |n\rangle\langle m|. \tag{3.5}$$

One can regard Φ_{nm} in eq. (3.5) as components of an $N \times N$ matrix Φ in the $N \to \infty$ limit. We then get the following relation [1, 2, 14]:

$$\text{Any field on NC } \mathbf{R}^{2n} \;\cong\; N \times N \text{ matrix at } N \to \infty. \tag{3.6}$$

If $\widehat{\Phi}$ is a real field, then Φ should be a Hermitian matrix. The relation (3.6) means that a NC field can be regarded as a master field of a large N matrix.

We have to point out that our statements in the previous section about emergent geometries should be understood in the 'semi-classical' limit where the Moyal-Weyl commutator, $-i[\widehat{f}, \widehat{g}]_\star$, can be reduced to the Poisson bracket $\{f, g\}_\theta$. Now the very notion of a point in NC spaces such as eq. (3.1) is doomed but replaced by a state in $\mathcal{H}$. So the usual concept of geometry based on smooth manifolds would be replaced by a theory of operator algebra, e.g., NC geometry *à la* Connes [41], or a theory of deformation quantization *à la* Kontsevich [31]. Thus our next mission is how to lift our previous 'semi-classical' arguments to the full NC world. A nice observation to do this is that a NC algebra $\mathcal{A}_\theta$

generated by the NC coordinates (1.3) is mathematically equivalent to the one generated by the NC phase space (1.1).

In classical mechanics, the set of possible states of a system forms a Poisson manifold and the observables that we want to measure are smooth functions in $C^\infty(M)$, forming a commutative (Poisson) algebra. In quantum mechanics, the set of possible states is a Hilbert space $\mathcal{H}$ and the observables are self-adjoint operators acting on $\mathcal{H}$, forming a NC $\star$-algebra. Pleasingly, there are two paths to represent the NC algebra. One is the matrix mechanics where the observables are represented by matrices in an arbitrary basis in $\mathcal{H}$. The other is the deformation quantization where, instead of building a Hilbert space from a Poisson manifold and associating an algebra of operators to it, the quantization is understood as a deformation of the algebra of classical observables. We are only concerned with the algebra to deform the commutative product in $C^\infty(M)$ to a NC, associative product. Two approaches have one to one correspondence through the Weyl-Moyal map [1].

Similarly, there are two different realizations of the NC algebra $\mathcal{A}_\theta$. One is the "matrix representation" we already introduced in eq. (3.6). The other is to map the NC $\star$-algebra $\mathcal{A}_\theta$ to a differential algebra using the inner automorphism, a normal subgroup of the full automorphism group, in $\mathcal{A}_\theta$. We call it "geometric representation", which will be used in section3.2. The geometric representation is quite similar to the dynamical evolution of a system in the Heisenberg picture in which the time-evolution of dynamical variables is defined by the inner automorphism of the NC $\star$-algebra generated by the coordinates in eq. (1.1). Of course, the two representations of a NC field theory should describe an equivalent physics. Now we will apply these two pictures to NC field theories to see what the equivalence between them implies.

3.1 Matrix representation

First we apply the matrix representation (3.6) to NC U(1) gauge theory on $\mathbf{R}^D = \mathbf{R}^d_C \times \mathbf{R}^{2n}_{NC}$ where the d-dimensional commutative spacetime $\mathbf{R}^d_C$ will be taken with either Lorentzian or Euclidean signature.[12] We will be brief since most technical details could be found in [14]. We decompose D-dimensional coordinates X^M $(M = 1, \cdots, D)$ into d-dimensional commutative ones, denoted as z^μ $(\mu = 1, \cdots, d)$, and $2n$-dimensional NC ones, denoted as y^a $(a = 1, \cdots, 2n)$, satisfying the relation (3.1). Likewise, D-dimensional gauge fields $\widehat{A}_M(z, y)$ are also decomposed in a similar way

$$\begin{aligned}\widehat{D}_M &= \partial_M - i\widehat{A}_M(z, y) \equiv (\widehat{D}_\mu, \widehat{D}_a)(z, y) \\ &= (\widehat{D}_\mu, -i\kappa B_{ab}\widehat{\Phi}^b)(z, y)\end{aligned} \tag{3.7}$$

[12] The generalized Darboux theorem was proved in [5], stating that any m-dimensional generalized complex manifold, via a diffeomorphism and a B-field transformation, looks locally like the product of an open set in $\mathbf{C}^k$ with an open set in the standard symplectic space $(\mathbf{R}^{2m-2k}, \sum dq^i \wedge dp_i)$. The integer k is called the type of the generalized complex structure, which is not necessarily constant but may rather vary throughout the manifold — the jumping phenomenon. The type can jump up, but always by an even number. Here we will consider the situation where the type k is constant over the manifold.

where $\widehat{D}_\mu = \partial_\mu - i\widehat{A}_\mu(z,y)$ are covariant derivatives along $\mathbf{R}^d_C$ and $\widehat{\Psi}_a(z,y) \equiv \kappa B_{ab}\widehat{\Phi}^b(z,y) = B_{ab}\widehat{x}^b(z,y)$ are adjoint Higgs fields of mass dimension defined by the covariant coordinates (2.22).

Here, the matrix representation means that NC U(1) gauge fields $\widehat{\Xi}_M(z,y) \equiv (\widehat{A}_\mu, \widehat{\Psi}_a)(z,y)$ are represented as $N \times N$ matrices in the $N \to \infty$ limit as eq. (3.5), i.e.,

$$\widehat{\Xi}_M(z,y) = \sum_{n,m=1}^{\infty} (\Xi_M)_{nm}(z)\, |n\rangle\langle m|. \tag{3.8}$$

Note that the $N \times N$ matrices $\Xi_M(z) = (A_\mu, \Psi_a)(z)$ in eq. (3.8) are now regarded as gauge and Higgs fields in $\mathrm{U}(N \to \infty)$ gauge theory on d-dimensional commutative spacetime $\mathbf{R}^d_C$. One can then show that, adopting the matrix representation (3.8), the NC U(1) gauge theory on $\mathbf{R}^d_C \times \mathbf{R}^{2n}_{NC}$ is "exactly" mapped to the $\mathrm{U}(N \to \infty)$ Yang-Mills theory on d-dimensional spacetime $\mathbf{R}^d_C$

$$\begin{aligned} S_B &= -\frac{1}{4g_{YM}^2}\int d^D X (\widehat{F}_{MN} - B_{MN}) \star (\widehat{F}^{MN} - B^{MN}) \\ &= -\frac{(2\pi\kappa)^{\frac{4-d}{2}}}{2\pi g_s}\int d^d z \mathrm{Tr}\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\mu\Phi^a D^\mu\Phi^a - \frac{1}{4}[\Phi^a,\Phi^b]^2\right) \end{aligned} \tag{3.9}$$

where the matrix $B_{MN} = \begin{pmatrix} 0 & 0 \\ 0 & B_{ab} \end{pmatrix}$ is the background symplectic 2-form (3.1) of rank $2n$. For notational simplicity, we have hidden all constant metrics in eq. (3.9). Otherwise, we refer [14] for the general expression.

We showed before that U(1) gauge symmetry in NC spaces is actually a spacetime symmetry (diffeomorphisms generated by X vector fields satisfying $\mathcal{L}_X B = 0$) where the NC U(1) gauge transformation acts on the covariant derivatives in (3.7) as

$$\widehat{D}_M \to \widehat{D}'_M = \widehat{U}(X) \star \widehat{D}_M \star \widehat{U}(X)^{-1} \tag{3.10}$$

for any NC group element $\widehat{U}(X) \in \mathrm{U}(1)$. Indeed the idea that NC gauge symmetries are spacetime symmetries was discussed long ago by many people. An exposition of these works can be found in [2]. The gauge transformation (3.10) can be represented in the matrix representation (3.5). The gauge symmetry now acts as unitary transformations on the Fock space $\mathcal{H}$ which is denoted as $U_{\mathrm{cpt}}(\mathcal{H})$. This NC gauge symmetry $U_{\mathrm{cpt}}(\mathcal{H})$ is so large that $U_{\mathrm{cpt}}(\mathcal{H}) \supset \mathrm{U}(N)$ $(N \to \infty)$ [42]. The NC U(1) gauge transformations in eq. (3.10) are now transformed into $\mathrm{U}(N)$ gauge transformations on $\mathbf{R}^d_C$ (where we complete $U_{\mathrm{cpt}}(\mathcal{H})$ with $\mathrm{U}(N)$ in the limit $N \to \infty$) given by

$$(D_\mu, \Psi_a) \to (D_\mu, \Psi_a)' = \mathrm{U}(z)(D_\mu, \Psi_a)\mathrm{U}(z)^{-1} \tag{3.11}$$

for any group element $\mathrm{U}(z) \in \mathrm{U}(N)$. Thus a NC gauge theory in the matrix representation can be regarded as a large N gauge theory.

As was explained above, the equivalence bewteen a NC U(1) gauge theory in higher dimensions and a large N gauge theory in lower dimensions is an exact map. What is the physical consequence of this exact equivalence ?

Indeed one can get a series of matrix models from the NC U(1) gauge theory (3.9). For instance, the IKKT matrix model for $d = 0$ [38], the BFSS matrix model for $d = 1$ [43] and the matrix string theory for $d = 2$ [44]. The most interesting case is that the 10-dimensional NC U(1) gauge theory on $\mathbf{R}^4_C \times \mathbf{R}^6_{NC}$ is equivalent to the bosonic part of 4-dimensional $\mathcal{N} = 4$ supersymmetric U(N) Yang-Mills theory, which is the large N gauge theory of the AdS/CFT duality [24]. Note that all these matrix models or large N gauge theories are a nonperturbative formulation of string or M theories. Therefore it should not be so surprising that a D-dimensional gravity could be emergent from the d-dimensional U($N \to \infty$) gauge theory, according to the large N duality or AdS/CFT correspondence and thus from the D-dimensional NC gauge theory in eq. (3.9). We will show further evidences that the action (3.9) describes a theory of (quantum) gravity.

A few remarks are in order.

(1) The equivalence (3.9) raises a far-reaching question about the renormalization property of NC field theory. If we look at the first action in eq. (3.9), the theory superficially seems to be non-renormalizable for $D > 4$ since the coupling constant $g^2_{YM} \sim m^{4-D}$ has a negative mass dimension. But this non-renormalizability appears as a fake if we use the second action in eq. (3.9). The resulting coupling constant, denoted as $g^2_d \sim m^{4-d}$, in the matrix action (3.9) depends only on the dimension of the commutative spacetime rather than the entire spacetime [14].

The change of dimensionality is resulted from the relationship (3.6) where all dependence of NC coordinates appears as matrix degrees of freedom. An important point is that the NC space (1.3) now becomes an n-dimensional positive integer lattice (fibered n-torus $\mathbf{T}^n$, but whose explicit dependence is mysteriously not appearing in the matrix action (3.9)). Thus the transition from commutative to NC spaces accompanies the mysterious cardinality transition *à la* Cantor from aleph-one (real numbers) to aleph-null (natural numbers). Of course this transition is akin to that from classical to quantum world in quantum mechanics. The transition from a continuum space to a discrete space should be radical even affecting the renormalization property [45].

Actually the matrix regularization of a continuum theory is an old story, for instance, a relativistic membrane theory in light-front coordinates (see, for example, a review [46] and references therein). The matrix regularization of the membrane theory on a Riemann surface of any genus is based on the fact that the symmetry group of area-preserving diffeomorphisms can be approximated by U(N). This fact in turn alludes that adjoint fields in U(N) gauge theory should contain multiple branes with arbitrary topologies. In this sense it is natural to think of the matrix theory (3.9) as a second quantized theory from the point of view of the target space [46].

(2) From the above construction, we know that the number of adjoint Higgs fields Φ^a is equal to the rank of the B-field (3.1). Therefore the matrix theory in eq. (3.9) can be defined in different dimensions by changing the rank of the B-field. This change of dimensionality appears in the matrix theory as the 'matrix T-duality' (see section

JHEP05(2009)012

VI.A in [46]) defined by[13]

$$iD_\mu \rightleftarrows \Phi^a. \tag{3.12}$$

Applying the matrix T-duality (3.12) to the action (3.9), on one hand, one can arrive at the 0-dimensional IKKT matrix model (in the case of Euclidean signature) or the 1-dimensional BFSS matrix model (in the case of Lorentzian signature). On the other hand, one can also go up to D-dimensional pure U(N) Yang-Mills theory given by

$$S_C = -\frac{1}{4g_{YM}^2}\int d^D X \mathrm{Tr}\, F_{MN}F^{MN}. \tag{3.13}$$

Note that the B-field is now completely disappeared, i.e., the spacetime is commutative. In fact the T-duality between eq. (3.9) and eq. (3.13) is an analogue of the Morita equivalence on a NC torus stating that NC U(1) gauge theory with rational $\theta = M/N$ is equivalent to an ordinary U(N) gauge theory [22].

(3) One may notice that the second action in eq. (3.9) can also be obtained by a dimensional reduction of the action (3.13) from D-dimensions to d-dimensions. However there is a subtle but important difference between these two.

A usual boundary condition for NC gauge fields in eq. (3.9) is that $\widehat{F}_{MN} \to 0$ at $|X| \to \infty$. So the following maximally commuting matrices

$$[\Phi^a, \Phi^b] = 0 \quad \cong \quad \Phi^a = \mathrm{diag}(\phi_1^a, \cdots, \phi_N^a), \quad \forall a \tag{3.14}$$

could not be a vacuum solution of eq. (3.9) (see eq. (2.28)), while they could be for the Yang-Mills theory dimensionally reduced from eq. (3.13). The vacuum solution of eq. (3.9) is rather eq. (3.1).

A proper interpretation for the contrast will be that the flat space $\mathbf{R}^{2n}$ in eq. (3.9) is not *a priori* given but defined by (or emergent from) the background (3.1). (We will show this fact later.) But, in eq. (3.13), a flat D-dimensional spacetime $\mathbf{R}^D$ already exists, so it is no longer needed to specify a background for the spacetime, contrary to eq. (3.9). It was shown by Witten [47] that the low-energy theory describing a system of N parallel Dp-branes in flat spacetime is the dimensional reduction of $\mathcal{N} = 1$, (9+1)-dimensional super Yang-Mills theory to $(p+1)$ dimensions. The vacuum solution describing a condensation of N parallel Dp-branes in flat spacetime is then given by eq. (3.14). So a natural inference is that the condensation of N parallel Dp-branes in eq. (3.14) is described by a different class of vacua from the background (3.1).

[13] One can change the dimensionality of the matrix model by any integer number by the matrix T-duality (3.12) while the rank of the B-field can be changed only by an even number. Hence it is not obvious what kind of background can explain the NC field theory with an odd number of adjoint Higgs fields. A plausible guess is that there is a 3-form $C_{\mu\nu\rho}$ which reduces to the 2-form B in eq. (3.1) by a circle compactification, so may be of M-theory origin. Unfortunately, we don't know how to construct a corresponding NC field theory with the 3-form background, although very recent developments seem to go toward that direction.

JHEP05(2009)012

3.2 Geometric representation

Now we move onto the geometric representation of a NC field theory. A crux is that translations in NC directions are an inner automorphism of the NC $\star$-algebra $\mathcal{A}_\theta$ generated by the coordinates in eq. (3.1),

$$e^{-ik^a B_{ab} y^b} \star \widehat{f}(z,y) \star e^{ik^a B_{ab} y^b} = \widehat{f}(z, y+k) \tag{3.15}$$

for any $\widehat{f}(z,y) \in \mathcal{A}_\theta$. Its infinitesimal form defines the inner derivation (1.4) of the algebra $\mathcal{A}_\theta$. It might be worthwhile to point out that the inner automorphism (3.15) is nontrivial only in the case of a NC algebra. In other words, commutative algebras do not possess any inner automorphism. In addition, eq. (3.15) clearly shows that (finite) space translations are equal to (large) gauge transformations.[14] It is a generic feature in NC spaces that an internal symmetry of physics turns into a spacetime symmetry, as we already observed in eq. (2.23).

If electromagnetic fields are present in the NC space (3.1), covariant objects, e.g., eq. (3.7), under the NC U(1) gauge transformation should be introduced. As an innocent generalization of the inner automorphism (3.15), let us consider the following "dynamical" inner automorphism

$$e^{k^M \widehat{D}_M} \star \widehat{f}(X) \star e^{-k^M \widehat{D}_M} = \widehat{W}(X, C_k) \star \widehat{f}(X+k) \star \widehat{W}(X, C_k)^{-1} \tag{3.17}$$

where

$$e^{k^M \widehat{D}_M} \equiv \widehat{W}(X, C_k) \star e^{k^M \partial_M} \tag{3.18}$$

with $\partial_M \equiv (\partial_\mu, -iB_{ab}y^b)$ and we used eqs.(3.15) and (3.16) which can be summarized with a compact form

$$e^{k^M \partial_M} \star \widehat{f}(X) \star e^{-k^M \partial_M} = \widehat{f}(X+k). \tag{3.19}$$

To understand eq. (3.17), first notice that $e^{k^M \widehat{D}_M}$ is a covariant object under NC U(1) gauge transformations according to eq. (3.10) and so one can get

$$\begin{aligned} e^{k^M \widehat{D}_M} \to e^{k^M \widehat{D}'_M} &= \widehat{U}(X) \star e^{k^M \widehat{D}_M} \star \widehat{U}(X)^{-1} \\ &= \widehat{U}(X) \star \widehat{W}(X, C_k) \star \widehat{U}(X+k)^{-1} \star e^{k^M \partial_M} \end{aligned} \tag{3.20}$$

[14] It may be interesting to compare with a similar relation on a commutative space

$$e^{l^\mu \partial_\mu} f(z,y) e^{-l^\mu \partial_\mu} = f(z+l, y). \tag{3.16}$$

A crucial difference is that translations in commutative space are an outer automorphism since $e^{l^\mu \partial_\mu}$ is not an element of the underlying $\star$-algebra. So every points in commutative space are distinguishable, i.e., unitarily inequivalent while every "points" in NC space are indistinguishable, i.e., unitarily equivalent. As a result, one loses the meaning of "points" in NC spacetime. This is a consequence of the fact that the set of prime ideals defining the spectrum of the algebra $\mathcal{A}_\theta$ is rather small for $\theta \neq 0$ contrary to the commutative case. Note that, after turning on $\hbar$, the relation (3.16) turns into an inner automorphism of NC algebra generated by the NC phase space (1.1) since $e^{l^\mu \partial_\mu} = e^{\frac{i}{\hbar} l^\mu p_\mu}$ is now an algebra element. Another intriguing difference is that the translation in (3.16) is parallel to the generator ∂_μ while the translation in (3.15) is transverse to the generator y^a due to the antisymmetry of B_{ab}. It would be interesting to contemplate this fact from the perspective in the footnote 3.

where eq. (3.19) was used. eq. (3.20) indicates that $\widehat{W}(X, C_k)$ is an extended object whose extension is proportional to the momentum k^M. Indeed $\widehat{W}(X, C_k)$ is the open Wilson line, well-known in NC gauge theories, defined by

$$\widehat{W}(X, C_k) = P_\star \exp\Big(i \int_0^1 d\sigma \partial_\sigma \xi^M(\sigma) \widehat{A}_M(X + \xi(\sigma))\Big), \tag{3.21}$$

where $P_\star$ denotes path ordering with respect to the $\star$-product along the path C_k parameterized by

$$\xi^M(\sigma) = k^M \sigma. \tag{3.22}$$

The most interesting feature in NC gauge theories is that there do not exist local gauge invariant observables in position space as eq. (3.15) shows that the 'locality' and the 'gauge invariance' cannot be compatible simultaneously in NC space. Instead NC gauge theories allow a new type of gauge invariant observables which are nonlocal in position space but localized in momentum space. These are the open Wilson lines in eq. (3.21) and their descendants with arbitrary local operators attached at their endpoints. It turns out [48] that these nonlocal gauge invariant operators behave very much like strings ! Indeed this behavior might be expected from the outset since both theories carry their own non-locality scales set by α' (string theory) and θ (NC gauge theories) which are equally of dimension of $(\text{length})^2$, as advertised in the table 1.

The inner derivation (1.4) in the presence of gauge fields is naturally covariantized by considering an infinitesimal version of the dynamical inner automorphism (3.17)[15]

$$\begin{aligned} ad_{\widehat{D}_A}[\widehat{f}](X) &\equiv [\widehat{D}_A, \widehat{f}]_\star(X) = D_A^M(z,y)\frac{\partial f(X)}{\partial X^M} + \cdots \\ &\equiv D_A[f](X) + \mathcal{O}(\theta^3), \end{aligned} \tag{3.23}$$

where $D_A^\mu = \delta_A^\mu$ since we define $[\partial_\mu, \widehat{f}(X)]_\star = \frac{\partial f(X)}{\partial z^\mu}$. It is easy to check that the covariant inner derivation (3.23) satisfies the Leibniz rule and the Jacobi identity, i.e.,

$$ad_{\widehat{D}_A}[\widehat{f} \star \widehat{g}] = ad_{\widehat{D}_A}[\widehat{f}] \star \widehat{g} + \widehat{f} \star ad_{\widehat{D}_A}[\widehat{g}], \tag{3.24}$$

$$\big(ad_{\widehat{D}_A} \star ad_{\widehat{D}_B} - ad_{\widehat{D}_B} \star ad_{\widehat{D}_A}\big)[\widehat{f}] = ad_{[\widehat{D}_A, \widehat{D}_B]_\star}[\widehat{f}]. \tag{3.25}$$

In particular, one can derive from eq. (3.25) the following identities

$$ad_{[\widehat{D}_A, \widehat{D}_B]_\star}[\widehat{f}](X) = -i[\widehat{F}_{AB}, \widehat{f}]_\star(X) = [D_A, D_B][f](X) + \cdots \tag{3.26}$$

$$[ad_{\widehat{D}_A}, [ad_{\widehat{D}_B}, ad_{\widehat{D}_C}]_\star]_\star[\widehat{f}](X) = -i[\widehat{D}_A \widehat{F}_{BC}, \widehat{f}]_\star(X) \equiv \mathfrak{R}_{ABC}{}^M(X)\partial_M f(X) + \cdots . \tag{3.27}$$

Note that the ellipses in the above equations correspond to higher order derivative corrections generated by generalized vector fields $\widehat{D}_A$.

We want to emphasize that the leading order of the map (3.23) is nothing but the Poisson algebra. It is well-known [32] that, for a given Poisson algebra $(C^\infty(M), \{\cdot, \cdot\}_\theta)$,

[15] From now on, for our later purpose, we denote the indices carried by the covariant objects in eq. (3.7) with $A, B, \cdots$ to distinguish them from those in the local coordinates X^M. The indices $A, B, \cdots$ will be raised and lowered using the flat Lorentzian metric η^{AB} and η_{AB}.

there exists a natural map $C^\infty(M) \to TM : f \mapsto X_f$ between smooth functions in $C^\infty(M)$ and vector fields in TM such that

$$X_f(g) = \{g, f\}_\theta \tag{3.28}$$

for any $g \in C^\infty(M)$. Indeed the assignment between a Hamiltonian function f and the corresponding Hamiltonian vector field X_f is the Lie algebra homomophism in the sense

$$X_{\{f,g\}_\theta} = -[X_f, X_g] \tag{3.29}$$

where the right-hand side represents the Lie bracket between the Hamiltonian vector fields. One can see that the Hamiltonian vector fields on M are the limit where the star-commutator $-i[\widehat{D}_A, \widehat{f}]_\star$ is replaced by the Poisson bracket $\{D_A, f\}_\theta$ or the Lie derivative $\mathcal{L}_{D_A}(f)$.

The properties, (3.24) and (3.25), show that the adjoint action (3.23) can be identified with the derivations of the NC algebra $\mathcal{A}_\theta$, which naturally generalizes the notion of vector fields. In addition their dual space will generalize that of 1-forms. Noting that the above NC differential algebra recovers the ordinary differential algebra at the leading order of NC deformations, it should be obvious that almost all objects known from the ordinary differential geometry find their counterparts in the NC case; e.g., a metric, connection, curvature and Lie derivatives, and so forth. Actually, according to the Lie algebra homomorphism (3.29), $D_A(X) = D_A^M(X)\frac{\partial}{\partial X^M}$ in the leading order of the map (3.23) can be identified with ordinary vector fields in TM where M is any D-dimensional (pseudo-)Riemannian manifold. More precisely, the D-dimensional NC U(1) gauge fields $\widehat{D}_M(X) = (\widehat{D}_\mu, \widehat{D}_a)(X)$ at the leading order appear as vector fields (frames in tangent bundle) on a D-dimensional manifold M given by

$$D_\mu(X) = \partial_\mu + A_\mu^a(X)\frac{\partial}{\partial y^a}, \qquad D_a(X) = D_a^b(X)\frac{\partial}{\partial y^b}, \tag{3.30}$$

where

$$A_\mu^a \equiv -\theta^{ab}\frac{\partial \widehat{A}_\mu}{\partial y^b}, \qquad D_a^b \equiv \delta_a^b - \theta^{bc}\frac{\partial \widehat{A}_a}{\partial y^c}. \tag{3.31}$$

Thus the map in eq. (3.23) definitely leads to the vector fields

$$D_A(X) = (\partial_\mu + A_\mu^a \partial_a, D_a^b \partial_b) \tag{3.32}$$

or with matrix notation[16]

$$D_A^M(X) = \begin{pmatrix} \delta_\mu^\nu & A_\mu^a \\ 0 & D_a^b \end{pmatrix}. \tag{3.33}$$

[16]We notice that this structure shares a striking similarity with the Kaluza-Klein construction of non-Abelian gauge fields from a higher dimensional Einstein gravity [49]. (Our matrix convention is swapping the row and column in [49].) We will discuss in section 5 a possible origin of the similarity between the Kaluza-Klein theory and the emergent gravity. A very similar Kaluza-Klein type origin of gravity from NC gauge theory was also noticed in the earlier work [8] where it was shown that a particular reduction of NC gauge theory captures the qualitative manner in which NC gauge transformations realize general covariance.

One can easily check from eq. (3.31) that D_A's in eq. (3.32) take values in the Lie algebra of volume-preserving vector fields, i.e., $\partial_M D^M_A = 0$. One can also determine the dual basis $D^A = D^A_M dX^M \in T^*M$ defined by eq. (A.1) which is given by

$$D^A(X) = \big(dz^\mu, V^a_b(dy^b - A^b_\mu dz^\mu)\big) \tag{3.34}$$

or with matrix notation

$$D^A_M(X) = \begin{pmatrix} \delta^\nu_\mu & -V^a_b A^b_\mu \\ 0 & V^a_b \end{pmatrix} \tag{3.35}$$

where $V^c_a D^b_c = \delta^b_a$.

Through the dynamical inner automorphism (3.17), NC U(1) gauge fields $\widehat{A}_M(X)$ or U($N \to \infty$) gauge-Higgs system (A_μ, Φ^a) in the action (3.9) are mapped to vector fields in TM (to be general "a NC tangent bundle" TM_θ) defined by eq. (3.23). This is a remarkably transparent way to get a D-dimensional gravity emergent from NC gauge fields or large N gauge fields. We provide in appendix A a rigorous proof of the equivalence between self-dual NC electromagnetism and self-dual Einstein gravity, originally first shown in [12], to illuminate how the map (3.23) achieves the duality between NC gauge fields and Riemannian geometry.

Now our next goal is obvious; the emergent gravity in general. Since the equation of motion (A.34) for self-dual NC gauge fields is mapped to the Einstein equation (A.22) for self-dual four-manifolds, one may anticipate that the equations of motion for arbitrary NC gauge fields would be mapped to the vacuum Einstein equations, in other words,

$$\widehat{D}^A \widehat{F}_{AB} = 0 \quad \stackrel{?}{\Longleftrightarrow} \quad E_{MN} \equiv R_{MN} - \frac{1}{2} g_{MN} R = 0 \tag{3.36}$$

together with the Bianchi identities

$$\widehat{D}_{[A} \widehat{F}_{BC]} = 0 \quad \stackrel{?}{\Longleftrightarrow} \quad R_{M[ABC]} = 0. \tag{3.37}$$

(We will often use the notation $\Gamma_{[ABC]} = \Gamma_{ABC} + \Gamma_{BCA} + \Gamma_{CAB}$ for the cyclic permutation of indices.) After some thought one may find that the guess (3.36) is not a sound reasoning since it should be implausible if arbitrary NC gauge fields allow only Ricci flat manifolds. Furthermore we know well that the NC U(1) gauge theory (3.9) will recover the usual Maxwell theory in the commutative limit. But if eq. (3.36) is true, the Maxwell has been lost in the limit. Therefore we conclude that the guess (3.36) must be something wrong.

We need a more careful musing about the physical meaning of emergent gravity. The emergent gravity proposes to take Einstein gravity as a collective phenomenon of gauge fields living in NC spacetime much like the superconductivity in condensed matter physics where it is understood as a collective phenomenon of Cooper pairs (spin-0 bound states of two electrons). It means that the origin of gravity is the collective excitations of NC gauge fields at scales $\sim l^2_{nc} = |\theta|$ which are described by a new order parameter, probably of spin-2, and they should be responsible to gravity even at large scales $\gg l_{nc}$, like as the classical physics emerges as a coarse graining of quantum phenomena when $\hbar \ll 1$ (the correspondence principle). Therefore the emergent gravity presupposes a spontaneous symmetry breaking of some big symmetry (see the table 3) to trigger a spin-2 order parameter (graviton as a Cooper pair of two gauge fields). If any, "the correspondence principle" for the emergent gravity will be that it should recover the Maxwell theory (possibly with some other fields) coupling to the Einstein gravity in commutative limit $|\theta| \to 0$ or at large distance scales $\gg l_{nc}$.[17] Then the Maxwell theory will appear in the right-hand side of the Einstein equation as an energy-momentum tensor, i.e.,

$$E_{MN} = \frac{8\pi G_D}{c^4} T_{MN} \tag{3.38}$$

where G_D is the gravitational Newton constant in D dimensions.

Let us first discuss the consequence of the gravitational correspondence principle postponing to section 5 the question about the existence of spin-2 bound states in NC spacetime. According to the above scheme, we are regarding the NC U(1) gauge theory in eq. (3.9) as a theory of gravity. Hence the parameters, g_{YM} and $|\theta|$, defining the NC gauge theory should be related to the gravitational Newton constant G_D defining the emergent gravity in D dimensions. A dimensional analysis (recovering $\hbar$ and c too) simply shows that

$$\frac{G_D \hbar^2}{c^2} \sim g_{YM}^2 |\mathrm{Pf}\theta|^{\frac{1}{n}} \tag{3.39}$$

where $2n$ is the rank of θ^{ab}. Suppose that g_{YM} is nonzero and always $c = 1$ in eq. (3.39). One can take a limit $|\theta| \to 0$ and $\hbar \to 0$ simultaneously such that G_D is nonzero. In this limit we will get the classical Einstein gravity coupling with the Maxwell theory which we are interested in. Instead one may take a limit $|\theta| \to 0$ and $G_D \to 0$ simultaneously, but $\hbar \neq 0$. This limit will correspond to quantum electrodynamics. On the other hand, the classical Maxwell theory will correspond to the limit, $\frac{G_D\hbar^2}{|\mathrm{Pf}\theta|^{\frac{1}{n}}} \sim g_{YM}^2 = \text{constant}$, when $G_D \to 0$, $\hbar \to 0$ and $|\theta| \to 0$.[18]

We will check the above speculation by showing that eq. (3.38) is correct equations of motion for emergent gravity. Indeed we will find the Einstein gravity with the energy-momentum tensor given by Maxwell fields and a "Liouville" field related to the volume factor in eq. (3.48). But we will see that the guess (3.37) is generally true. Note that self-dual gauge fields have a vanishing energy-momentum tensor that is the reason why the self-dual NC gauge fields simply satisfy the relation in eq. (3.36).

We will use the notation in appendix A with obvious minor changes for a D-dimensional Lorentzian manifold. Define structure equations of the vectors $D_A \in TM$ as

$$[D_A, D_B] = -\mathfrak{f}_{AB}{}^C D_C \tag{3.40}$$

[17]This is not to say that the electromagnetism is only relevant to the emergent gravity. The weak and the strong forces should play a role in some way which we don't know yet. But we guess that they will affect only a microscopic structure of spacetime since they are short range forces. See section 4 for some related discussion.

[18]As a completely different limit, one may keep $|\theta|$ nonzero while $g_{YM} \to 0$. Note that this limit does not necessarily mean that NC gauge theories are non-interacting since, for an adjoint scalar field $\widehat{\phi}$ as an example, $\widehat{D}_a\widehat{\phi} = \partial_a\widehat{\phi} - i\frac{g_{YM}}{\hbar c}[\widehat{A}_a, \widehat{\phi}]_\star = \partial_a\widehat{\phi} + \frac{g_{YM}\theta^{bc}}{\hbar c}\frac{\partial\widehat{A}_a}{\partial y^b}\frac{\partial\widehat{\phi}}{\partial y^c} + \cdots$, recovering the original form of gauge coupling. $\frac{g_{YM}\theta^{bc}}{\hbar c}$ can be nonzero depending on the limit under control. The relation (3.39) implies that there exist gravitational ($G_D \neq 0$) and non-gravitational ($G_D = 0$) theories for the case at hand. Unfortunately we did not understand what they are.

JHEP05(2009)012

where $\mathfrak{f}_{AB}{}^{\mu} = 0, \ \forall A, B$ for the basis (3.32). From the experience of the self-dual case, we know that the vector fields D_A are related to the orthonormal frames (vielbeins) E_A by $D_A = \lambda E_A$ where the conformal factor λ will be determined later. (This situation is reminiscent of the string frame (D_A) and the Einstein frame (E_A) in string theory.) Hence the D-dimensional metric is given by

$$\begin{aligned} ds^2 &= \eta_{AB} E^A \otimes E^B \\ &= \lambda^2 \eta_{AB} D^A \otimes D^B = \lambda^2 \eta_{AB} D^A_M D^B_N \, dX^M \otimes dX^N \end{aligned} \tag{3.41}$$

where $E^A = \lambda D^A$. In particular, the dual basis (3.34) determines its explicit form up to a conformal factor as [50]

$$ds^2 = \lambda^2 \Big(\eta_{\mu\nu} dz^\mu dz^\nu + \delta_{ab} V^a_c V^b_d (dy^c - \mathbf{A}^c)(dy^d - \mathbf{A}^d) \Big) \tag{3.42}$$

where $\mathbf{A}^a = A^a_\mu dz^\mu$. The structure function $\mathfrak{f}_{AB}{}^C$ is also conformally mapped to eq. (A.11) with

$$\mathfrak{f}_{AB}{}^C = \lambda f_{AB}{}^C - D_A \log \lambda \delta^C_B + D_B \log \lambda \delta^C_A. \tag{3.43}$$

In the case of $D = 4$, eq. (3.26) immediately shows that the leading order of self-dual NC gauge fields described by eq. (A.34) reduces to the following self-duality equation

$$\mathfrak{f}_{AB}{}^E = \pm \frac{1}{2} \varepsilon_{AB}{}^{CD} \mathfrak{f}_{CD}{}^E. \tag{3.44}$$

We proved in appendix A that the metric (3.42) satisfying eq. (3.44) describes self-dual Einstein manifolds where the conformal factor λ^2 is given by eq. (A.32).

Now let us fix the conformal factor λ^2 in the metric (3.41). By an $SO(d-1,1) \times SO(2n)$ rotation of basis vectors E_A, we can impose the condition that

$$f_{BA}{}^B \equiv \phi_A = (3 - D) E_A \log \lambda \tag{3.45}$$

and eq. (3.43) in turn implies

$$\mathfrak{f}_{BA}{}^B \equiv \rho_A = 2 D_A \log \lambda. \tag{3.46}$$

Note that $\mathfrak{f}_{AB}{}^{\mu} = 0, \ \forall A, B$ which is the reason why one has to use only $SO(d-1,1) \times SO(2n)$ rotations to achieve the condition (3.45) (see the footnote 23 for a similar argument for self-dual gauge fields). eq. (3.45) means that the vector fields E_A are volume preserving with respect to a D-dimensional volume form $\mathfrak{v} = \lambda^{(3-D)} \mathfrak{v}_g$ where

$$\mathfrak{v}_g = E^1 \wedge \cdots \wedge E^D \tag{3.47}$$

and then the vector fields D_A are volume preserving with respect to the volume form $\mathfrak{v}_D = \lambda^{(2-D)} \mathfrak{v}_g$. (See eq. (A.31) for its proof.) Therefore we get[19]

$$\lambda^2 = \mathfrak{v}_D (D_1, \cdots, D_D). \tag{3.48}$$

[19] One can directly check eq. (3.46) as follows. Acting $\mathcal{L}_{D_A}$ on both sides of eq. (3.48), we get $\mathcal{L}_{D_A} \Big(\mathfrak{v}_D (D_1, \cdots, D_D) \Big) = (\mathcal{L}_{D_A} \mathfrak{v}_D)(D_1, \cdots, D_D) + \sum_{B=1}^D \mathfrak{v}_D (D_1, \cdots, \mathcal{L}_{D_A} D_B, \cdots, D_D) = (\mathcal{L}_{D_A} \mathfrak{v}_D)(D_1, \cdots, D_D) + \sum_{B=1}^D \mathfrak{v}_D (D_1, \cdots, [D_A, D_B], \cdots, D_D) = (\nabla \cdot D_A + \mathfrak{f}_{BA}{}^B) \mathfrak{v}_D (D_1, \cdots, D_D) = (2 D_A \log \lambda) \mathfrak{v}_D (D_1, \cdots, D_D)$. Since $\mathcal{L}_{D_A} \mathfrak{v}_D = (\nabla \cdot D_A) \mathfrak{v}_D = 0$, eq. (3.46) is deduced. Conversely, if $\mathfrak{f}_{BA}{}^B = 2 D_A \log \lambda$, D_A's all preserve the volume form $\mathfrak{v}_D$, i.e., $\mathcal{L}_{D_A} \mathfrak{v}_D = (\nabla \cdot D_A) \mathfrak{v}_D = 0$.

Since $\partial_M D_A^M = 0$, we know that the invariant volume is given by $\mathfrak{v}_D = dz^1 \wedge \cdots \wedge dz^d \wedge dy^1 \wedge \cdots \wedge dy^{2n}$. Therefore we finally get

$$\lambda^2 = \det{}^{-1} V_b^a. \tag{3.49}$$

In terms of the structure functions one can get the map in eq. (3.27)

$$-i[\widehat{D}_A \widehat{F}_{BC}, \widehat{f}]_\star = \left(D_A \mathfrak{f}_{BC}{}^D - \mathfrak{f}_{BC}{}^E \mathfrak{f}_{AE}{}^D\right) D_D[f] + \cdots . \tag{3.50}$$

In other words, one can get the following maps for the equations of motion and the Bianchi identities

$$\widehat{D}^A \widehat{F}_{AB} = 0 \quad \Longleftrightarrow \quad \eta^{AB}\Big(D_A \mathfrak{f}_{BC}{}^D - \mathfrak{f}_{BC}{}^E \mathfrak{f}_{AE}{}^D\Big) = 0, \tag{3.51}$$

$$\widehat{D}_{[A} \widehat{F}_{BC]} = 0 \quad \Longleftrightarrow \quad D_{[A} \mathfrak{f}_{BC]}{}^D - \mathfrak{f}_{[BC}{}^E \mathfrak{f}_{A]E}{}^D = 0. \tag{3.52}$$

The spacetime geometry described by the metric (3.41) or (3.42) is an emergent gravity arising from NC gauge fields whose underlying theory is defined by the action (3.9). The fundamental variables in our approach are of course gauge fields which should be subject to eqs.(3.51) and (3.52). A spacetime metric is defined by NC (or non-Abelian) gauge fields and regarded as a collective variable (a composite or bilinear of gauge fields). Therefore our goal is to show that the equations of motion (3.51) for NC gauge fields together with the Bianchi identity (3.52) can be rewritten using the map (3.23) as the Einstein equation for the metric (3.41). In other words, the Einstein equation $E_{MN} = 8\pi G_D T_{MN}$ is nothing but the equation of motion for NC gauge fields represented from the (emergent) spacetime point of view.

Our strategy is the following. First note that the Riemann curvature tensors defined by eq. (B.6) have been expressed with the orthonormal basis E_A. Since we will impose on them eqs.(3.51) and (3.52), it will be useful to represent them with the gauge theory basis D_A. As a consequence, it will be shown that Einstein manifolds emerge from NC gauge fields after imposing eqs.(3.51) and (3.52). All calculations can straightforwardly be done using the relations (3.43) and (B.10). All the details show up in appendix B.

The result is very surprising. The emergent gravity derived from NC gauge fields predicts a new form of energy which we call the "Liouville" energy-momentum tensor. Indeed this form of energy was also noticed in [17] with a nonvanishing Ricci scalar. The terminology is attributed to the following fact. The vector fields D_A are volume preserving with respect to the symplectic volume $\mathfrak{v}_D$ (see the footnote 19). Thus $\mathfrak{v}_D$ is constant along integral curves of D_A, in which case D_A are called incompressible with respect to $\mathfrak{v}_D$ and which is known as the *Liouville theorem* in Hamiltonian mechanics [32]. (See [30] for the Liouville theorem in curved spacetime.) Superficially this seems to imply that spacetime behaves like an incompressible fluid so that spacetime volume does not change along the flow generated by the vector field D_A. But we have to be careful to interpret the geometrical meaning of the Liouville theorem because the symplectic volume $\mathfrak{v}_D$ is different from the Riemannian volume $\mathfrak{v}_g = \lambda^{(D-2)}\mathfrak{v}_D$ in eq. (3.47). Furthermore, as we showed in appendix B, the vector field D_A contributes to both sides of the Einstein equation (3.38).

So the spacetime volume given by $\mathfrak{v}_g$ can change along the flow described by the vector field D_A and its shape may also change in very complicated ways. But this kind of a local expansion, distortion and twisting of spacetime manifold will spend some energy, which should be supplied from the right-hand side. This picture may be clarified by looking at the so-called Raychaudhuri equation [51, 52].

The Raychaudhuri equation is evolution equations of the expansion, shear and rotation of flow lines along the flow generated by a vector field in a background spacetime. Here we introduce an affine parameter τ labeling points on the curves of the flow. Given a timelike unit vector field u^M, i.e., $u^M u_M = -1$, the Raychaudhuri equation in D dimensions is given by

$$\dot{\Theta} - \dot{u}^M_{;M} + \Sigma_{MN}\Sigma^{MN} - \Omega_{MN}\Omega^{MN} + \frac{1}{D-1}\Theta^2 = -R_{MN}u^M u^N. \tag{3.53}$$

$\Theta = u^M{}_{;M}$ represents the expansion/contraction of volume and $\dot{\Theta} = \frac{d\Theta}{d\tau}$ while $\dot{u}^M = u^M{}_{;N}u^N$ represents the acceleration due to nongravitational forces, e.g., the Lorentz force. Σ_{MN} and Ω_{MN} are the shear tensor and the vorticity tensor, respectively, which are all orthogonal to u^M, i.e., $\Sigma_{MN}u^N = \Omega_{MN}u^N = 0$. The Einstein equation (3.38) can be rewritten as

$$R_{MN} = 8\pi G_D\big(T_{MN} - \frac{1}{2}g_{MN}T_P{}^P\big) \tag{3.54}$$

where $T_{MN} = E^A_M E^B_N T_{AB}$. In four dimensions, one can see from eq. (3.54) that the right-hand side of eq. (3.53) is given by

$$-R_{MN}u^M u^N = -\frac{1}{2\lambda^2}u^M u^N(\rho_M\rho_N + \Psi_M\Psi_N) - 8\pi G_4 T^{(M)}_{MN}u^M u^N \tag{3.55}$$

where the Lorentzian energy-momentum tensor in eq. (3.54) can be read off from eq. (B.37) and eq. (B.38) having in mind the footnote 26.

Suppose that all the terms except the expansion evolution $\dot{\Theta}$ on the left-hand side of eq. (3.53) as well as the Maxwell term $T^{(M)}_{MN}$ in eq. (3.55) vanish or become negligible. In this case the Raychaudhuri equation reduces to

$$\dot{\Theta} = -\frac{1}{2\lambda^2}u^M u^N(\rho_M\rho_N + \Psi_M\Psi_N). \tag{3.56}$$

Note that the Ricci scalar is given by $R = \frac{1}{2\lambda^2}g^{MN}(\rho_M\rho_N + \Psi_M\Psi_N)$. Therefore $R < 0$ when ρ_M and Ψ_M are timelike while $R > 0$ when ρ_M and Ψ_M are spacelike. Remember that our metric signature is $(-+++)$. So, for the timelike perturbations, $\dot{\Theta} < 0$ which means that the volume of a three dimensional spacelike hypersurface orthogonal to u_M decreases. However, if spacelike perturbations are dominant, the volume of the three dimensional spacelike hypersurface can expand. For example, consider the most symmetric perturbations as in eq. (B.50), i.e.,

$$\langle\rho_A\rho_B\rangle = \frac{1}{4}\eta_{AB}\rho_C^2, \quad \langle\Psi_A\Psi_B\rangle = \frac{1}{4}\eta_{AB}\Psi_C^2. \tag{3.57}$$

More precisely, one can decompose the perturbation (3.56) into trace (scalar), anti-symmetric (vector) and symmetric-traceless (tensor) parts. Since we look at only the scalar perturbation in eq. (3.53), simply assume that the vector and tensor modes are negligible for some reasons, e.g., the cosmological principle. In this case, eq. (3.56) becomes

$$\dot{\Theta} = \frac{1}{8\lambda^2} g^{MN}(\rho_M \rho_N + \Psi_M \Psi_N) > 0. \tag{3.58}$$

The perturbation (3.57) does not violate the energy condition since $u^M u^N T^{(L)}_{MN} = \frac{1}{64\pi G_4 \lambda^2} g^{MN}(\rho_M \rho_N + \Psi_M \Psi_N) > 0$. See eq. (3.95). This means that the spacetime geometry is in a de Sitter phase. Thus we see that the Liouville energy-momentum tensor can act as a source of gravitational repulsion. We will further discuss in section 3.4 this energy as a plausible candidate of dark energy.

Up to now we have considered fluctuations around the vacuum (3.1) corresponding to a uniform condensation of gauge fields. In this case if we turn off all fluctuations, i.e., $\widehat{A}_M = 0$ in eq. (3.23), the metric (3.41) or (3.42) simply reduces to a flat spacetime. We have to point out that the fluctuations need not be small. Our ignorance of the next leading order, $\mathcal{O}(\theta^3)$, in eq. (3.23) corresponds to the limit of slowly varying fields, $\sqrt{2\pi\alpha'}|\frac{\partial F}{F}| \ll 1$, in the sense keeping field strengths (without restriction on their size) but not their derivatives [3]. Since the Ricci curvature (B.27) is purely determined by $\mathfrak{f}_{ABC} \sim F_{AB}$ (see eq. (B.39)), this approximation corresponds to the limit of slowly varying curvatures compared to the NC scale $|\theta| \sim l_{nc}^2$ but without restriction on their size. This implies that NC effects should be important for a violently varying spacetime, e.g., near the curvature singularity, as expected.

3.3 General NC spacetime

Now the question is how to generalize the emergent gravity picture to the case of a nontrivial vacuum, e.g., eq. (2.26), describing an inhomogeneous condensate of gauge fields. The Poisson structure $\Theta^{ab}(x) = (\frac{1}{B})^{ab}(x)$ is nonconstant in this case, so the corresponding NC field theory is defined by a nontrivial star-product

$$[Y^a, Y^b]_{\tilde{\star}} = i\Theta^{ab}(Y) \tag{3.59}$$

where Y^a denote vacuum coordinates which are designed with the capital letters to distinguish them from y^a for the constant vacuum (3.1). The star product $[\widehat{f}, \widehat{g}]_{\tilde{\star}}$ for $\widehat{f}, \widehat{g} \in \mathcal{A}_\Theta$ can be perturbatively computed via the deformation quantization [31]. There are excellent earlier works [53] especially relevant for the analysis of the DBI action as a generalized geometry though a concrete formulation of NC field theories for a general NC spacetime is still out of reach.

Recall that we are interested in the commutative limit so that

$$\begin{aligned} -i[\widehat{f}, \widehat{g}]_{\tilde{\star}} &= \Theta^{ab}(Y)\frac{\partial f(Y)}{\partial Y^a}\frac{\partial g(Y)}{\partial Y^b} + \cdots \\ &\equiv \{f, g\}_\Theta + \cdots \end{aligned} \tag{3.60}$$

for $\widehat{f}, \widehat{g} \in \mathcal{A}_\Theta$. Using the Poisson bracket (3.60), we can similarly realize the Lie algebra homomophism $C^\infty(M) \to TM : f \mapsto X_f$ between a Hamiltonian function f and the corresponding Hamiltonian vector field X_f. To be specific, for any given function $f \in$

$C^\infty(M)$, we can always assign a Hamiltonian vector field X_f defined by $X_f(g) = \{g, f\}_\Theta$ with some function $g \in C^\infty(M)$. Then the following Lie algebra homomophism holds

$$X_{\{f,g\}_\Theta} = -[X_f, X_g] \tag{3.61}$$

as long as the Jacobi identity for the Poisson bracket $\{f, g\}_\Theta$ holds or, equivalently, the Schouten-Nijenhuis bracket for the Poisson structure Θ^{ab} vanishes [31].

Furthermore there is a natural automorphism $D(\hbar)$ which acts on star-products [31]:

$$f \tilde{\star} g = D(\hbar)\Big(D(\hbar)^{-1}(f) \star D(\hbar)^{-1}(g)\Big). \tag{3.62}$$

In the commutative limit where $D(\hbar) \approx 1$, eq. (3.62) reduces to the following condition

$$\{f, g\}_\Theta = \{f, g\}_\theta. \tag{3.63}$$

Let us explain what eq. (3.63) means. For $f = Y^a(y)$ and $g = Y^b(y)$, eq. (3.63) implies that

$$\Theta^{ab}(Y) = \theta^{cd}\frac{\partial Y^a}{\partial y^c}\frac{\partial Y^b}{\partial y^d} \tag{3.64}$$

whose statement is, of course, equivalent to the Moser lemma (2.13). Also notice that eq. (3.63) defines diffeomorphisms between vector fields $X'_f(g) \equiv \{g, f\}_\Theta$ and $X_f(g) \equiv \{g, f\}_\theta$ such that

$$X'^{\,a}_f = \frac{\partial Y^a}{\partial y^b}X^b_f. \tag{3.65}$$

Indeed the automorphism (3.62) corresponds to a global statement that the two star-products involved are cohomologically equivalent in the sense that they generate the same Hochschild cohomology [31].

It is still premature to know the precise form of the full NC field theory defined by the star product (3.60). Even the commutative limit where the star commutator reduces to the Poisson bracket in eq. (3.60) still bears some difficulty since the derivatives of Θ^{ab} appear here and there. For example,

$$\{B_{ab}(Y)Y^b, f\}_\Theta = \frac{\partial f}{\partial Y^a} + \Theta^{bc}\frac{\partial B_{ad}}{\partial Y^b}Y^d\frac{\partial f}{\partial Y^c}. \tag{3.66}$$

In particular, $\{B_{ab}(Y)Y^b, f\}_\Theta \neq \frac{\partial f}{\partial Y^a}$. There is no simple way to realize the derivative $\frac{\partial}{\partial Y^a}$ as an inner derivation.[20] Now we will suggest an interesting new approach for the nontrivial background (2.26) based on the remark (3) in section 2.3.

Let us return to the remark (3). Denote the nontrivial B-field in eq. (2.26) as

$$B_{ab}(x) = (\bar{B} + \bar{F}(x))_{ab} \tag{3.67}$$

[20]To be precise, we have to point out that the extra term in eq. (3.66) can be ignored under the limit of our consideration. We are considering the limit of slowly varying fields where the derivative of field strengths is ignored (see the last paragraph in section 3.2). Then eq. (3.66) defines the inner derivation in this limit. We expect the analysis in this limit will be very straightforward. But we will not push to this direction because the coming new approach seems to provide a more clear insight for the emergent geometry.

where $\bar{B}_{ab} = \left(\theta^{-1}\right)_{ab}$ describes a constant background such as eq. (3.1) while $\bar{F}(x) = d\bar{A}(x)$ describes an inhomogeneous condensate of gauge fields. Then the left-hand side of eq. (2.24) is of the form $g + \kappa(\bar{B} + \mathcal{F})$ where $\mathcal{F} = d\mathcal{A}$ with $\mathcal{A}(x) = \bar{A}(x) + A(x)$. It should be completely conceivable that it can be mapped to the NC gauge theory of the gauge field $\mathcal{A}(x)$ in the constant $\bar{B}$-field background according to the Seiberg-Witten equivalence [22]. Let us denote the corresponding NC gauge field as $\widehat{A}_a \equiv \widehat{B}_a + \widehat{C}_a$. The only notable point is that the gauge field $\widehat{A}_a$ has an inhomogeneous background part $\widehat{B}_a$ and $\widehat{C}_a$ describes fluctuations around this background. This situation should be familiar, for example, with a gauge theory in an instanton (or soliton) background.

So everything goes parallel to the previous case. We will suppose a general situation so that the background gauge fields $\widehat{\bar{A}}_\mu(z,y)$ as well as $\widehat{B}_b(z,y)$ depend on z^μ. Let us introduce the following covariant coordinates

$$\begin{aligned} \widehat{X}^a(z,y) &= y^a + \theta^{ab}\widehat{A}_b(z,y) = y^a + \theta^{ab}\widehat{B}_b(z,y) + \theta^{ab}\widehat{C}_b(z,y) \\ &\equiv Y^a(z,y) + \theta^{ab}\widehat{C}_b(z,y) \end{aligned} \tag{3.68}$$

where we identified the vacuum coordinates Y^a in eq. (3.59) because we have to recover them after completely turning off the fluctuation $\widehat{C}_b$. Now the covariant derivative $\widehat{D}_M$ in eq. (3.7) can be defined in the exactly same way

$$\widehat{D}_M = \partial_M - i\widehat{A}_M(z,y) = (\widehat{D}_\mu, -i\bar{B}_{ab}\widehat{X}^b)(z,y) \tag{3.69}$$

where $\partial_M = (\partial_\mu, -i\bar{B}_{ab}y^b)$. In addition the NC fields $\widehat{D}_A$ in eq. (3.69) (see the footnote 15) can be mapped to vector fields in the same way as eq. (3.23).

Since the results in section 3.2 can be applied to arbitrary NC gauge fields in the constant B-field, the same formulae can be applied to the present case at hand with the understanding that the vector fields D_A in eq. (3.23) refer to total gauge fields including the background. This means that the vector fields $D_A = \lambda E_A \in TM$ reduce to $\bar{D}_A = \bar{\lambda}\bar{E}_A$ after completely turning off the fluctuations where $\bar{D}_A$ is determined by the background $(\partial_\mu - i\widehat{\bar{A}}_\mu(z,y), -i\bar{B}_{ab}Y^b(z,y))$ and $\bar{\lambda}$ satisfies the relation

$$\bar{\lambda}^2 = \mathfrak{v}_D(\bar{D}_1, \cdots, \bar{D}_D). \tag{3.70}$$

Therefore the metric for the background is given by

$$\begin{aligned} ds^2 &= \eta_{AB}\bar{E}^A \otimes \bar{E}^B \\ &= \bar{\lambda}^2\eta_{AB}\bar{D}^A \otimes \bar{D}^B = \bar{\lambda}^2\eta_{AB}\bar{D}^A_M\bar{D}^B_N\, dX^M \otimes dX^N. \end{aligned} \tag{3.71}$$

Of course we have implicitly assumed that the background $\bar{D}_A$ also satisfies eqs.(3.51)-(3.52). In four dimensions, for instance, we know that the metric (3.71) describes Ricci-flat four manifolds if $\bar{D}_A$ satisfies the self-duality equation (3.44).

Now let us look at the picture of the right-hand side of eq. (2.24). After applying the Darboux transform (2.13) for the symplectic structure (3.67), the right-hand side becomes of the form $h_{ab}(y) + \kappa(\bar{B}_{ab} + \mathfrak{F}_{ab}(y))$ where

$$\mathfrak{F}_{ab}(y) = \frac{\partial x^\alpha}{\partial y^a}\frac{\partial x^\beta}{\partial y^b}F_{\alpha\beta}(x) \equiv \partial_a\mathfrak{A}_b(y) - \partial_b\mathfrak{A}_a(y) \tag{3.72}$$

JHEP05(2009)012

and the metric $h_{ab}(y)$ is given by eq. (2.25). Note that in this picture the gauge fields $\mathfrak{A}_a(y)$ are regarded as fluctuations propagating in the background $h_{ab}(y)$ and $\bar{B}_{ab}$. Therefore it would be reasonable to interpret the right-hand side of eq. (2.24) as a NC gauge theory of the gauge field $\mathfrak{A}_a(y)$ defined by the canonical NC space (3.1) but in curved space described by the metric $h_{ab}(y)$.

Although the formulation of NC field theory in a generic curved spacetime is still a challenging problem, we want to speculate on how to formulate the emergent gravity within this picture since the underlying picture for the identity (2.24) is rather transparent. In this regard, the results in [53] would be useful. In this approach the inhomogeneous condensate of gauge fields in the vacuum (3.67) appears as an explicit background metric, which implies that the metric (3.41) in this picture will be replaced by

$$\begin{aligned} ds^2 &= g_{AB} E^A \otimes E^B \\ &= \Lambda^2 g_{AB} D^A \otimes D^B = \Lambda^2 g_{AB} D^A_M D^B_N \, dX^M \otimes dX^N \end{aligned} \tag{3.73}$$

where g_{AB} is the metric in the space spanned by the noncoordinate bases $E^A = \Lambda D^A$ [49]. Since the anholonomic basis D^A in eq. (3.73) is supposed to be flat when the fluctuations are turned off, i.e., $\mathfrak{F}_{ab} = 0$, the metric $\Lambda^2 g_{AB}$ will correspond to the background metric $h_{ab}(y)$ in the DBI action (2.24). Since the metric (3.73) has the Riemannian volume form $\mathfrak{v}_g = \sqrt{-g} E^1 \wedge \cdots \wedge E^D$ instead of eq. (3.47), the volume form $\mathfrak{v}_D = \Lambda^{(2-D)} \mathfrak{v}_g$ in eq. (3.48) will be given by

$$\mathfrak{v}_D = \sqrt{-g} \Lambda^2 D^1 \wedge \cdots \wedge D^D. \tag{3.74}$$

So the function Λ in eq. (3.73) will satisfy the condition

$$\sqrt{-g} \Lambda^2 = \mathfrak{v}_D(D_1, \cdots, D_D). \tag{3.75}$$

And it is easy to infer that $\sqrt{-g}\Lambda^2 \to 1$ for vanishing fluctuations since D_A becomes flat for that case.

According to the metric (3.73), the indices $A, B, \cdots$ will be raised and lowered using the metric g_{AB}. As usual, the torsion free condition (B.3) for the metric (3.73) will be imposed to get the relation (B.4) where $\omega_{ABC} = g_{BD} {\omega_A}^D{}_C$ and $f_{ABC} = g_{CD} {f_{AB}}^D$. Since g_{AB} is not a flat metric, ${\omega_A}^B{}_C$ in eq. (B.1) or eq. (B.2) will actually be the Levi-Civita connections in noncoordinate bases rather than the spin connections, but we will keep the notation for convenience. And the condition that the metric (3.73) is covariantly constant, i.e., $\nabla_C \Big(g_{AB} E^A \otimes E^B \Big) = 0$, leads to the relation [49]

$$\omega_{ABC} = \frac{1}{2} \big(E_A g_{BC} - E_B g_{CA} + E_C g_{AB} \big) + \frac{1}{2} \big(f_{ABC} - f_{BCA} + f_{CAB} \big). \tag{3.76}$$

The curvature tensors have exactly the same form as eq. (B.6).

All the calculations in appendix B can be repeated in this case although the details will be much more complicated. We will not perform this calculation since it seems to be superfluous at this stage. But we want to draw some interesting consequences from the natural requirement that the metric (3.73) must be equivalent to the metric (3.41) or (3.42) in general, not only for backgrounds.

Let us summarize the two pictures we have employed. Let us indicate the first picture with (L) and the second picture with (R). When all fluctuations are vanishing, we have the following results:

$$\begin{aligned}(\mathrm{L}): ds^2 &= \bar{\lambda}^2\eta_{AB}\bar{D}^A_M\bar{D}^B_N\, dX^M\otimes dX^N \\ &= \bar{\lambda}^2\Big(\eta_{\mu\nu}dz^\mu dz^\nu + \delta_{ab}V^a_cV^b_d(dy^c-\mathbf{A}^c)(dy^d-\mathbf{A}^d)\Big) \qquad (3.77)\\ \mathfrak{v}_D &= dz^1\wedge\cdots\wedge dz^d\wedge dy^1\wedge\cdots\wedge dy^{2n} \qquad (3.78)\\ \bar{\lambda}^2 &= \det{}^{-1}V^a_b \qquad (3.79)\\ (\mathrm{R}): ds^2 &= \Lambda^2 g_{MN}dX^M\otimes dX^N \qquad (3.80)\\ \mathfrak{v}_D &= dz^1\wedge\cdots\wedge dz^d\wedge dy^1\wedge\cdots\wedge dy^{2n} \qquad (3.81)\\ \Lambda^2 &= \frac{1}{\sqrt{-g}}. \qquad (3.82)\end{aligned}$$

One can immediately see that (L) and (R) are equal each other if $g_{MN}=\eta_{AB}\bar{D}^A_M\bar{D}^B_N$. Indeed, this equivalence is nothing but the geometric manifestation of the equivalence (2.24). Therefore we conjecture that the equivalence between the two pictures (L) and (R) remains true even after including all fluctuations.

Now let us examine whether the action (3.9) allows a conformally flat metric as a solution. First we point out that $\Lambda^2=1$ for the flat metric $g_{MN}=\eta_{MN}$ as eq. (3.82) immediately shows. This can also be seen from the picture (L). Since we put $\mathbf{A}^c=0$, $g_{MN}=\eta_{MN}$ corresponds to a coordinate transformation $y^a\to\tilde{y}^a$ such that $V^a_bdy^b=d\tilde{y}^a$. This coordinate transformation can be expressed as $D^b_a=\frac{\partial y^b}{\partial\tilde{y}^a}$ using eq. (3.31). That is, the coordinate $\tilde{y}^a$ is a solution of the equation $D_a\tilde{y}^b\equiv\frac{\partial\tilde{y}^b}{\partial y^a}+\{\widehat{A}_a,\tilde{y}^b\}_\theta=\delta^b_a$. Thus we can replace the vector field $D_a\in TM$ by $\frac{\partial}{\partial\tilde{y}^a}$ in the space described by the coordinates $(z^\mu,\tilde{y}^a)$. Then eq. (3.70) is automatically satisfied since the volume form (3.78) is equal to $\mathfrak{v}_D=\det{}^{-1}V^a_b\; dz^1\wedge\cdots\wedge dz^d\wedge d\tilde{y}^1\wedge\cdots\wedge d\tilde{y}^{2n}=\bar{\lambda}^2dz^1\wedge\cdots\wedge dz^d\wedge d\tilde{y}^1\wedge\cdots\wedge d\tilde{y}^{2n}$. Because we already put $\widehat{A}_\mu=0$, the vector fields in TM are now represented by $D_A[f](z^\mu,\tilde{y}^a)=\big(\frac{\partial}{\partial z^\mu},\frac{\partial}{\partial\tilde{y}^a}\big)[f]$, which implies $\forall\ \mathfrak{f}_{AB}{}^C=0$. Therefore $\bar{\lambda}$ should be a constant due to the relation (3.46).

Thereby we see that the conformally flat metric is instead given by the vector field $\bar{D}_A=\phi(z,y)\partial_A$, which corresponds to the coordinate transformations $z^\mu\to\tilde{z}^\mu$, $y^a\to\tilde{y}^a$ such that $dz^\mu=\phi^{-1}d\tilde{z}^\mu$ and $V^a_bdy^b=\phi^{-1}d\tilde{y}^a$. In this case the metric (3.77) and the volume form (3.78) are given by

$$\begin{aligned} ds^2 &= \phi^{D-2}\big(\eta_{\mu\nu}d\tilde{z}^\mu d\tilde{z}^\nu + d\tilde{y}^a d\tilde{y}^a\big) \qquad (3.83)\\ \mathfrak{v}_D &= d\tilde{z}^1\wedge\cdots\wedge d\tilde{z}^d\wedge d\tilde{y}^1\wedge\cdots\wedge d\tilde{y}^{2n} \qquad (3.84)\end{aligned}$$

where we used eq. (3.82), i.e., $\Lambda^2=\bar{\lambda}^2=\phi^D$. For the vector field $\bar{D}_A=\phi(\tilde{z},\tilde{y})\partial_A$, the equation of motion (3.51) becomes

$$0=\{\widehat{D}^A\widehat{F}_{AB},f\}_\theta=\phi(\partial^A\phi\partial_A\phi+\phi\partial^A\partial_A\phi)\partial_Bf-\phi(\partial^A\phi\partial_B\phi+\phi\partial^A\partial_B\phi)\partial_Af \qquad (3.85)$$

for any reference function $f=f(\tilde{z},\tilde{y})$.

We will try two kinds of simple ansatz

$$(I): \quad \phi = \phi(\tau) \ \text{ where } \ \tau = \tilde{z}^0, \tag{3.86}$$

$$(II): \quad \phi = \phi(\rho) \ \text{ where } \ \rho^2 = \sum_{a=1}^{2n} \tilde{y}^a \tilde{y}^a. \tag{3.87}$$

One can find for the ansatz (I) that eq. (3.85) leads to the equation $\frac{d}{d\tau}\big(\phi\frac{d\phi}{d\tau}\big) = 0$ and so $\phi(\tau) = \gamma\sqrt{\tau+\tau_0}$. In four dimensions, this solution describes an expanding cosmological solution [30, 52]. It is interesting that the expanding cosmological solution comes out from "pure" NC electromagnetism (3.9) without any source term.[21]

However, for the ansatz (II), we found that only $\phi =$ constant can be a solution. This seems to be true in general. Hence we claim that a conformally flat metric for the ansatz (II) is trivial. A source term might be added to the action (3.9) to realize a nontrivial solution. The solution for the ansatz (II) should be interesting because the $AdS_p \times \mathbf{S}^q$ space with $q+1 = 2n$ belongs to this class and it can be described by eq. (3.83) by choosing

$$\phi^{D-2} = \frac{L^2}{\rho^2}. \tag{3.88}$$

In particular, $AdS_5 \times \mathbf{S}^5$ space is given by the case, $d = 4, \ n = 3$, that is,

$$ds^2 = \frac{L^2}{\rho^2}\big(\eta_{\mu\nu} d\tilde{z}^\mu d\tilde{z}^\nu + d\tilde{y}^a d\tilde{y}^a\big) = \frac{L^2}{\rho^2}\big(\eta_{\mu\nu} d\tilde{z}^\mu d\tilde{z}^\nu + d\rho^2\big) + L^2 d\Omega_5^2. \tag{3.89}$$

We hope to address in the near future what kind of source term should be added to get the conformal factor (3.88). eq. (3.88) looks like a potential of codimension-$2n$ Coulomb sources in D dimensions when we identify the harmonic function $H(\rho)^{\frac{1}{n-1}} = \phi^{D-2} = L^2/\rho^2$, which presumably corresponds to the vacuum (3.14).

3.4 Hindsights

We want to ponder on the spacetime picture revealed from NC gauge fields and the emergent gravity we have explored so far.

The most remarkable picture emerging from NC gauge fields is about the origin of flat spacetime, which is absent in Einstein gravity. Of course the notorious problem for emergent time is elusive as ever. We will refer to the emergence of spaces only here, but we will discuss in section 4 how "Emergent Time" would be defined in the context of emergent gravity.

Note that the flat spacetime is a geometry of special relativity rather than general relativity and the special relativity is a theory about kinematics rather than dynamics. Hence the general relativity says nothing about the dynamical origin of flat spacetime

[21] In comoving coordinates, the metric (3.83) is of the form $ds^2 = -dt^2 + a(t)^2 d\mathbf{x}^2$ where $t = \frac{2}{3}\gamma\tau^{\frac{3}{2}}$ and $a(t)^2 = \gamma^2\tau \equiv \alpha t^{\frac{2}{3}}$. Since $a(t) \propto t^{\frac{2}{3(1+w)}}$, we see that this metric corresponds to a universe characterized by the equation of state $p = \rho$, i.e., $w = 1$. It has been argued in [54] that the $p = \rho$ cosmology corresponds to the most holographic background and the most entropic initial condition for the universe. We thank Qing-Guo Huang for drawing our attention to [54].

since the flat spacetime defining a local inertial frame is assumed to be *a priori* given without reference to its dynamical origin. So there is a blind point about the dynamical origin of spacetime in general relativity.

Our scheme for the emergent gravity implies that the uniform condensation of gauge fields in a vacuum (3.1) will be a source of flat spacetime. Now we will clarify the dynamical origin of flat spacetime based on the geometric representation in section 3.2. We will equally refer to the commutative spacetime $\mathbf{R}^d_C$ with the understanding that it has been T-dualized from a fully NC space (except time) in the sense of eq. (3.12) although the transition from NC to commutative ones is mysterious (see the remark (1) in section 3.1). Therefore we will regard ∂_μ in eq. (3.23) as a background part since it is related to y^a/κ via the matrix T-duality (3.12).

The basic principle for the emergent gravity is the map (3.23) or the correspondence (3.28) between NC fields in $\mathcal{A}_\theta$ and vector fields in TM. The most notable point is that we necessarily need a Poisson (or symplectic) structure on M, viz., NC spacetime, to achieve the correspondence between $\mathcal{A}_\theta$ and $\Gamma(TM)$, sections of tangent bundle $TM \to M$. Basically the θ-deformation (1.3) introduces the duality between NC gauge fields and spacetime geometry. The crux is that there exists a novel form of the equivalence principle, guaranteed by the global Moser lemma, for the electromagnetism in the context of symplectic geometry. In this correspondence a flat spacetime is coming from the constant background itself defining the NC spacetime (3.1). This observation, trivial at the first glance, was the crucial point for the proposal in [15] to resolve the cosmological constant problem.

We know that the uniform condensation of stress-energy in a vacuum will appear as a cosmological constant in Einstein gravity. For example, if we shift a matter Lagrangian $\mathcal{L}_M$ by a constant Λ, that is,

$$\mathcal{L}_M \to \mathcal{L}_M - 2\Lambda, \tag{3.90}$$

this shift results in the change of the energy-momentum tensor of matter by $T_{MN} \to T_{MN} - \Lambda g_{MN}$ in the Einstein equation (3.38) although the equations of motion for matters are invariant under the shift [21]. Definitely this Λ-term will appear as a cosmological constant in Einstein gravity and it has an observable physical effect. For example, a flat spacetime is no longer a solution of the Einstein equation in the case of $\Lambda \neq 0$.

The emergent gravity defined by the action (3.9) responds completely differently to the constant shift (3.90). To be specific, let us consider a constant shift of the background $B_{MN} \to B_{MN} + \delta B_{MN}$. Then the action (3.9) in the new background becomes

$$S_{B+\delta B} = S_B + \frac{1}{2g_{YM}^2}\int d^D X \widehat{F}_{MN}\delta B_{MN} - \frac{1}{4g_{YM}^2}\int d^D X\Big(\delta B_{MN}^2 - 2B^{MN}\delta B_{MN}\Big). \tag{3.91}$$

The last term in eq. (3.91) is simply a constant and thus it will not affect the equations of motion (3.51). The second term is a total derivative and so it will vanish if $\widehat{F}_{MN}$ well behaves at infinity. (It is a defining property in the definition of a star product that $\int d^D X \widehat{f} \star \widehat{g} = \int d^D X \widehat{f} \cdot \widehat{g}$. Then the second term should vanish as far as $\widehat{A}_M \to 0$ at infinity.) If spacetime has a nontrivial boundary, the second term could be nonvanishing at the boundary which will change the theory under the shift. We will not consider a nontrivial spacetime boundary since the boundary term is not an essential issue in the cosmological constant problem, though there would be an interesting physics at the boundary. Then we get the result $S_{B+\delta B} \cong S_B$. Indeed this is the Seiberg-Witten equivalence between NC field theories defined by the noncommutativity $\theta' = \frac{1}{B+\delta B}$ and $\theta = \frac{1}{B}$ [22]. Although the vacuum (3.1) readjusts itself under the shift, the Hilbert spaces $\mathcal{H}_{\theta'}$ and $\mathcal{H}_\theta$ in eq. (3.2) are completely isomorphic if and only if θ and θ' are nondegenerate constants. Furthermore the vector fields in eq. (3.23) generated by $B + \delta B$ and B backgrounds are equally flat as long as they are constant. We also observed in eq. (B.44) that the background gauge field does not contribute to the energy-momentum tensor.

Therefore we conclude that the constant shift of energy density such as eq. (3.90) is a symmetry of the theory (3.9) although the action (3.9) defines a theory of gravity in the sense of emergent gravity. Thus the emergent gravity is completely immune from the vacuum energy. In other words, *the vacuum energy does not gravitate* unlike as Einstein gravity. This was an underlying logic in [15] why the emergent gravity can resolve the cosmological constant problem.

One has realized that the cosmological constant can be interpreted as a measure of the energy density of the vacuum. One finds that the resulting energy density is of the form

$$\rho_{\rm vac} = \frac{1}{V}\sum_{\mathbf{k}} \frac{1}{2}\hbar\omega_{\mathbf{k}} \sim \hbar k^4_{max} \tag{3.92}$$

where k_{max} is a certain momentum cutoff below which an underlying theory can be trusted. Thus the vacuum energy (3.92) may be understood as a vast accumulation of harmonic oscillators in space. Note that the vacuum (3.1) is also the uniform condensation of harmonic oscillators in space. The immune difference is that the harmonic oscillator in eq. (3.92) is defined by the NC phase space (1.1) while the harmonic oscillator in eq. (3.1) is defined by the NC space (1.3).

The current framework of quantum field theory, which has been confirmed by extremely sophisticated experiments, mostly predicts the vacuum energy of the order $\rho_{\rm vac} \sim (10^{18} GeV)^4$. The real problem is that this huge energy couples to gravity in the framework of Einstein gravity and so results in a bizarre contradiction with contemporary astronomical observations. This is the notorious cosmological constant problem.

But we have observed that the emergent gravity shows a completely different picture about the vacuum energy. The vacuum energy (3.92) does not gravitate regardless of how large it is as we explained above. So there is no cosmological constant problem in emergent gravity. More remarkable picture in emergent gravity is that the huge energy $M_{Pl} = (8\pi G)^{-1/2} \sim 10^{18} GeV$ is actually the origin of the flat spacetime. Here the estimation of the vacuum energy for the condensate (3.1), for example, $\rho_{\rm vac} \sim |B_{ab}|^2 \sim M_P^4$ in four dimensions, is coming from our identification of the Newton constant (3.39). In other words, the emergent gravity says that a flat spacetime is not free gratis but a result of the Planck energy condensation in a vacuum.

An important point is that the vacuum (3.1) triggered by the Planck energy condensation causes the spacetime to be NC and the NC spacetime is the essence of emergent gravity. Since the flat spacetime is emergent from the uniform vacuum (3.1) and the Lorentz symmetry is its spacetime symmetry, the dynamical origin of flat spacetime implies that the Lorentz symmetry is also emergent from the NC spacetime (3.1). In addition, if the vacuum (3.1) was triggered by the Planck energy condensation, the flat spacetime as well as the Lorentz symmetry should be very robust against any perturbations since the Planck energy is the maximum energy in Nature.

Furthermore the noble picture about the dynamical origin of the flat spacetime may explain why gravity is so weak compared to other forces. Let us look at eq. (2.22). As we know, y^a is a background part defining a flat spacetime and the gauge field $\widehat{A}_a$ describes dynamical fluctuations around the flat spacetime. (As we mentioned at the beginning of this section, the commutative space in eq. (3.7) can also be incorporated into this picture using the T-duality (3.12).) One may imagine these fluctuations as shaking the background spacetime lattice defined by the Fock space (3.2), which generates gravitational fields. But the background lattice is very solid since the stiffness of the lattice is supposed to be the Planck scale. In other words, the gravity generated by the deformations of the spacetime lattice (3.2) will be very weak since it is suppressed by the background stiffness of the Planck scale. So, ironically, the weakness of gravitational force may be due to the fact that the flat spacetime is originated from the Planck energy.

The emergent gravity thus reveals a remarkably beautiful and consistent picture about the origin of flat spacetime. Does it also say something about dark energy ?

Over the past ten or twenty years, several magnificent astronomical observations have confirmed that our Universe is composed of 5 % ordinary matters and radiations while 23 % dark matter and 72 % dark energy. The observed value of the dark energy turned out to be very very tiny, say,

$$\Delta\rho^{obs} \leq (10^{-12} GeV)^4 \tag{3.93}$$

which is desperately different from the theoretical estimation (3.92) by the order of 10^{120}. What is the origin of the tiny dark energy (3.93) ?

We suggested in [15] that the dark energy (3.93) is originated from vacuum fluctuations around the primary background (3.1). Since the background spacetime (3.1) is NC, any UV fluctuations of the Planck scale L_P in the NC spacetime will be necessarily paired with IR fluctuations of a typical scale L_H related to the size of cosmic horizon in our Universe due to the UV/IR mixing [55]. A simple dimensional analysis shows that the energy density of the vacuum fluctuation is of the order

$$\Delta\rho \sim \frac{1}{L_P^2 L_H^2} \tag{3.94}$$

which is numerically in agreement with the observed value (3.93) up to a factor [15]. It should be remarked that the vacuum fluctuation (3.94) will be an inevitable consequence if our picture about the dynamical origin of flat spacetime is correct. If the vacuum (3.1) or equivalently the flat spacetime is originated from the Planck energy condensation (it should be the case if the identification (3.39) is correct), the energy density of the vacuum (3.1) will be $\rho_{\rm vac} \sim M_{Pl}^4$ which is the conventionally identified vacuum energy predicted by quantum field theories. Thus it is natural to expect that cosmological fluctuations around the vacuum (3.1) or the flat spacetime will add a tiny energy $\Delta\rho$ to the vacuum so that the total energy density is equal to $\rho \sim M_{Pl}^4(1+\frac{L_P^2}{L_H^2})$ since $L_P^2 \equiv 8\pi G_4$ and $L_H^2 \equiv 1/\Lambda$ are only the relevant scales in the Einstein equation (3.38) with $T_{MN} = -\frac{\Lambda}{8\pi G_4} g_{MN} = -M_{Pl}^4 \big(\frac{L_P}{L_H}\big)^2 g_{MN}$ [21]. Since the first term does not gravitate, the second term (3.94) will be the leading contribution to the deformation of spacetime curvature, leading to possibly a de Sitter phase. It should be remarked that the fluctuation (3.94) is of the finite size L_H. So one cannot apply the argument (3.91) since $\Delta\rho$ is not constant over the entire spacetime even if it is constant over a Hubble patch.

Now we will argue that the Liouville energy (B.38) may (or can) explain the dark energy (3.94). First let us perform the Wick rotation for the energy-momentum tensor (B.38) using the rule in the footnote 26 to get the Lorentzian energy-momentum tensor in the 4-dimensional spacetime. It is then given by

$$T_{MN}^{(L)} = \frac{1}{16\pi G_4 \lambda^2}\Big(\rho_M\rho_N + \Psi_M\Psi_N - \frac{1}{2}g_{MN}(\rho_P^2+\Psi_P^2)\Big) \tag{3.95}$$

where $\rho_M = 2\partial_M\lambda$ and $\Psi_M = E_M^A\Psi_A$. First of all we emphasize that we already checked in eq. (3.56) that it can exert a negative pressure causing an expansion of universe, possibly leading to a de Sitter phase. We also pointed out below eq. (B.51) that it can behave like a cosmological constant, i.e., $\rho = -p$, in a constant (or almost constant) curvature spacetime. Another important property is that the Liouville energy (3.95) is vanishing for the flat spacetime. So it should be small if the spacetime is not so curved.

To be more quantitative, let us consider the fluctuation (3.57) and look at the energy density $u^M u^N T_{MN}^{(L)}$ along the flow represented by a timelike unit vector u^M as in eq. (3.55). Note that the Riemannian volume is given by $\mathfrak{v}_g = \lambda^2\mathfrak{v}_4 = \lambda^2 d^4x$. Also recall that Ψ_M is the Hodge-dual to the 3-form H in eq. (B.47). Thus $u^M\rho_M$ and $u^M\Psi_M$ refer to the volume change of a three dimensional spacelike hypersurface orthogonal to u^M. Assume that the radius of the three dimensional hypersurface is $R(\tau)$ at time τ, where τ is an affine parameter labeling the curve of the flow. Then it is reasonable to expect that $u^M\rho_M \approx u^M\Psi_M \approx 2\lambda/R(\tau)$ where we simply assumed that $u^M\rho_M \approx u^M\Psi_M$. Then we approximately get

$$u^M u^N T_{MN}^{(L)} \sim \frac{1}{8\pi G_4 R^2}. \tag{3.96}$$

If we identify the radius R with the size of cosmic horizon, L_H, the energy density (3.96) reproduces the dark energy (3.94) up to a factor.

4 Electrodynamics as a symplectic geometry

This section does contain mostly speculations. We will not intend any rigor. Rather we will revisit the $\hbar$-deformation (1.1) to reinterpret the electrodynamics of a charged particle in terms of symplectic geometry defined in phase space. We want to point out its beautiful aspects since in our opinion it has not been well appreciated by physicists. Furthermore it will provide a unifying view about U(1) gauge theory in terms of symplectic geometry. Nevertheless our main motivation for the revival is to get some glimpse on how to introduce matter fields within the framework of emergent gravity. As a great bonus, it will also outfit us with a valuable insight about how to define "Time" in the sense of emergent spacetime.

4.1 Hamiltonian dynamics and emergent time

Let us start to revisit the derivation of the Darboux theorem (2.13) due to Moser [34]. A remarkable point in the Moser's proof is that there always exists a one-parameter family of diffeomorphisms generated by a smooth time-dependent vector field X_t satisfying $\iota_{X_t}\omega_t + A = 0$ for the change of a symplectic structure within the same cohomology class from ω to $\omega_t = \omega + t(\omega' - \omega)$ for all $0 \leq t \leq 1$ where $\omega' - \omega = dA$. The evolution of the symplectic structure is locally described by the flow ϕ_t of X_t starting at $\phi_0 =$ identity. (Of course the "time" t here is just an affine parameter labeling the flow. At this stage it does not necessarily refer to a physical time.) By the Lie derivative formula, we have

$$\begin{aligned}\frac{d}{dt}(\phi_t^*\omega_t) &= \phi_t^*(\mathcal{L}_{X_t}\omega_t) + \phi_t^*\frac{d\omega_t}{dt} \\ &= \phi_t^* d\iota_{X_t}\omega_t + \phi_t^*(\omega' - \omega) = \phi_t^*(\omega' - \omega - dA) = 0.\end{aligned} \tag{4.1}$$

Thus $\phi_1^*\omega' = \phi_0^*\omega = \omega$, so ϕ_1 provides a chart describing the evolution from ω to $\omega' = \omega + dA$.

A whole point of the emergent gravity is the global existence of the one-parameter family of diffeomorphisms ϕ_t describing the local deformation of a symplectic structure due to the electromagnetic force. Therefore the electromagnetism in NC spacetime is nothing but a symplectic geometry (at the leading order or commutative limit). Now our question is how to understand matter fields or particles in the context of emergent geometry or symplectic geometry.

As a first step, we want to point out that the coupling of a charged particle with U(1) gauge fields is beautifully understood in the context of symplectic geometry [25, 26]. This time the symplectic geometry of matters is involved with the $\hbar$-deformation (1.1) rather than the θ-deformation (1.3) which is the symplectic geometry of gravity. It is rather natural that matters or particles are described by the symplectic geometry of the phase space since the particles by definition are prescribed by their positions and momenta besides their intrinsic charges, e.g., spin, electric charge, isospin, etc. We will consider only the electric charge among their internal charges for simplicity. We refer some interesting works [25, 26, 56, 57] addressing this problem.

Let (M, ω) be a symplectic manifold. One can properly choose local canonical coordinates $y^a = (q^1, p_1, \cdots, q^n, p_n)$ in M such that the symplectic structure ω can be written in the form

$$\omega = \sum_{i=1}^n dq^i \wedge dp_i. \tag{4.2}$$

Then $\omega \in \bigwedge^2 T^*M$ can be thought as a bundle map $TM \to T^*M$. Since ω is nondegenerate at any point $y \in M$, we can invert this map to obtain the map $\vartheta \equiv \omega^{-1} : T^*M \to TM$. This cosymplectic structure $\vartheta \in \bigwedge^2 TM$ is called the Poisson structure of M which defines a Poisson bracket $\{\cdot,\cdot\}_\vartheta$. See the footnote 6. In a local chart with coordinates y^a, we have

$$\{f,g\}_\vartheta = \sum_{a,b=1}^{2n} \vartheta^{ab} \frac{\partial f}{\partial y^a} \frac{\partial g}{\partial y^b}. \tag{4.3}$$

Let $H : M \to \mathbf{R}$ be a smooth function on a Poisson manifold M. The vector field X_H defined by $\iota_{X_H}\omega = dH$ is called the Hamiltonian vector field with the energy function H. We define a dynamical flow by the differential equation

$$\frac{df}{dt} = X_H(f) + \frac{\partial f}{\partial t} = \{f,H\}_\vartheta + \frac{\partial f}{\partial t}. \tag{4.4}$$

A solution of the above equation is a function f such that for any path $\gamma : [0,1] \to M$ we have

$$\frac{df(\gamma(t))}{dt} = \{f,H\}_\vartheta(\gamma(t)) + \frac{\partial f(\gamma(t))}{\partial t}. \tag{4.5}$$

The dynamics of a charged particle in an external static magnetic field is described by the Hamiltonian

$$H = \frac{1}{2m}\big(\mathbf{p} - e\mathbf{A}\big)^2 \tag{4.6}$$

which is obtained by the free Hamiltonian $H_0 = \frac{\mathbf{p}^2}{2m}$ with the replacement

$$\mathbf{p}' = \mathbf{p} - e\mathbf{A}. \tag{4.7}$$

Here the electric charge of an electron is $q_e = -e$ and e is a coupling constant identified with g_{YM}. The symplectic structure (4.2) leads to the Hamiltonian vector field X_H given by

$$X_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i}. \tag{4.8}$$

Then the Hamilton's equation (4.4) reduces to the well-known Lorentz force law

$$m\frac{d\mathbf{v}}{dt} = e\mathbf{v} \times \mathbf{B}. \tag{4.9}$$

An interesting observation [25] (orginally due to Jean-Marie Souriau) is that the Lorentz force law (4.9) can be derived by keeping the Hamiltonian $H = H_0$ but instead shifting the symplectic structure

$$\omega \to \omega' = \omega - eB \tag{4.10}$$

where $B(q) = \frac{1}{2}B_{ij}(q)dq^i \wedge dq^j$. In this case the Hamiltonian vector field X_H defined by $\iota_{X_H}\omega' = dH$ is given by

$$X_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \left(\frac{\partial H}{\partial q^i} - eB_{ij}\frac{\partial H}{\partial p_j}\right)\frac{\partial}{\partial p_i}. \tag{4.11}$$

Then one can easily check that the Hamilton's equation (4.4) with the vector field (4.11) reproduces the Lorentz force law (4.9). Actually one can show that the symplectic structure

ω' in eq. (4.10) introduces a NC phase space [1] such that the momentum space becomes NC, i.e., $[p'_i, p'_j] = -i\hbar e B_{ij}$.

If a particle is interacting with electromagnetic fields, the influence of the magnetic field $B = dA$ is described by the 'minimal coupling' (4.7) and the new momenta $\mathbf{p}' = -i\hbar(\nabla - i\frac{e}{\hbar}\mathbf{A})$ are covariant under U(1) gauge transformations. Let us point out that the minimal coupling (4.7) can be understood as the Darboux transformation (2.13) between ω and ω'. Consider the coordinate transformation $y^a \mapsto x^a(y) = (Q^1, P_1, \cdots, Q^n, P_n)(q,p)$ such that

$$\sum_{i=1}^n dq^i \wedge dp_i = \sum_{i=1}^n dQ^i \wedge dP_i - \frac{e}{2}\sum_{i,j=1}^n B_{ij}(Q) dQ^i \wedge dQ^j \tag{4.12}$$

but the Hamiltonian is unchanged, i.e., $H = \frac{\mathbf{P}^2}{2m}$. The condition (4.12) is equivalent to the following equations

$$\begin{aligned} \frac{\partial q^i}{\partial Q^j}\frac{\partial p_i}{\partial Q^k} - \frac{\partial q^i}{\partial Q^k}\frac{\partial p_i}{\partial Q^j} &= -eB_{jk}, \\ \frac{\partial q^i}{\partial Q^j}\frac{\partial p_i}{\partial P_k} - \frac{\partial q^i}{\partial P_j}\frac{\partial p_i}{\partial Q^k} &= \delta^k_j, \\ \frac{\partial q^i}{\partial P_j}\frac{\partial p_i}{\partial P_k} - \frac{\partial q^i}{\partial P_k}\frac{\partial p_i}{\partial P_j} &= 0. \end{aligned} \tag{4.13}$$

The above equations are solved by

$$q^i = Q^i, \qquad p_i = P_i + eA_i(Q). \tag{4.14}$$

In summary the dynamics of a charged particle in an electromagnetic field has two equivalent descriptions:

$$\left(H = \frac{(\mathbf{p} - e\mathbf{A})^2}{2m}, \omega\right)(q,p) \quad \cong \quad \left(H = \frac{\mathbf{P}^2}{2m}, \omega' = \omega - eB\right)(Q,P). \tag{4.15}$$

The equivalence (4.15) can easily be generalized to a time-dependent background $A^\mu = (A^0, \mathbf{A})(q,t)$ with the Hamiltonian $H = \frac{1}{2m}\big(\mathbf{p} - e\mathbf{A}\big)^2 + eA^0$. The Hamilton's equation (4.4) in this case becomes

$$m\frac{d\mathbf{v}}{dt} = e\big(\mathbf{E} + \mathbf{v} \times \mathbf{B}\big). \tag{4.16}$$

The equivalence (4.15) now means that the Lorentz force law (4.16) can be obtained by the Hamiltonian vector field (4.11) with the Hamiltonian $H = \frac{\mathbf{p}^2}{2m} + eA^0$ by noticing that the time dependence of the external fields now appears as the explicit t-dependence of momenta $p_i = p_i(t)$. Indeed the electric field $\mathbf{E}$ appears as the combination $\mathbf{E} = -\nabla A^0 + \frac{1}{e}\frac{\partial \mathbf{p}}{\partial t}$. But note that the coordinates (q^i, p_i) in eq. (4.11) correspond to (Q^i, P_i) in the notation (4.12) and so $\frac{\partial \mathbf{p}}{\partial t} = -e\frac{\partial \mathbf{A}}{\partial t}$ by eq. (4.14).

In a very charming paper [26], Dyson explains the Feynman's view about the electrodynamics of a charged particle. Feynman starts with an assumption that a particle exists with position q^i and velocity $\dot{q}_i$ satisfying commutation relations

$$[q^i, q^j] = 0, \qquad m[q^i, \dot{q}_j] = i\hbar\delta^i_j. \tag{4.17}$$

Then he asks a question: What is the most general form of forces appearing in the Newton's equation consistent with the commutation relation (4.17) ? Remarkably he ends up with the electromagnetic force (4.16). In a sense, the Feynman's result is a no-go theorem for the consistent interaction of particles in quantum mechanics. The only room for some modification to the Feynman's argument seems to introduce internal degrees of freedom such as spin, isospin, color, etc [56]. Then a particle motion is defined on $\mathbf{R}^3 \times F$ with an internal space F. The dynamics of the particle carrying an internal charge in F is defined by a symplectic structure on $T^*\mathbf{R}^3 \times F$. See [56] for some details.

The Feynman's approach clearly shows that the electromagnetism is an inevitable structure in quantum particle dynamics. Furthermore, as emphasized by Dyson, the Feynman's formulation shows that nonrelativistic Newtonian mechanics and relativistic Maxwell equations are coexisting peacefully. This is due to the gauge symmetry that the Lorentz force (4.16) is generated by the minimal coupling $p_\mu \to \mathfrak{P}_\mu \equiv p_\mu - eA_\mu$. Moreover, Souriau and Sternberg show that the minimal coupling can be encoded into the deformation of symplectic structure, which can be summarized as the relativistic form [57]: $\omega = -d\xi \to \omega' = \omega - eF = -d\big(\xi + eA\big)$ where $\xi = \mathfrak{P}_\mu dQ^\mu$ and $A = A_\mu(Q)dQ^\mu$. Therefore the Maxwell equation $dF = 0$ is simply interpreted as the closedness of the symplectic structure.

Now we have perceived that the dynamics of a charged particle can be interpreted as a symplectic geometry in phase space. The evolution of the system is described by the dynamical flow (4.5) generated by a Hamiltonian vector field, e.g., eq. (4.8), for a given Hamiltonian H. Basically, the time in the Hamilton's equation (4.4) is an affine parameter to trace out the history of a particle and it is operationally defined by the Hamiltonian. Therefore the time in the Hamiltonian dynamics is intrinsically assigned to the particle itself. But we have to notice that, only when the symplectic structure is fixed for a given Hamiltonian, the evolution of the system is completely determined by the evolution equation (4.4). In this case the dynamics of the system can be formulated in terms of an evolution with a single time parameter. In other words, we have a globally well-defined time for the evolution of the system. This is the usual situation we consider in classical mechanics.

We observed the equivalence (4.15) for the dynamics of a charged particle. Let us consider a dynamical evolution described by the change of a symplectic structure from ω to $\omega_t = \omega + t(\omega' - \omega)$ for all $0 \leq t \leq 1$ where $\omega' - \omega = -edA$. The Moser lemma (4.1) says that there always exists a one-parameter family of diffeomorphisms generated by a smooth time-dependent vector field X_t satisfying $\iota_{X_t}\omega_t = eA$. Although the vector field X_t defines a dynamical one-parameter flow, the vector field X_t is in general not even a locally Hamiltonian since $dA = B \neq 0$. The evolution of the system in this case is locally described by the flow ϕ_t of X_t starting at $\phi_0 =$ identity but it is no more a (locally) Hamiltonian flow. That is, there is no well-defined or global time for the particle system. The flow can be a (locally) Hamiltonian, i.e., $\phi_t =$ identity for all $0 \leq t \leq 1$, only for $dA = 0$. In other words, the time flow ϕ_t of X_t defined on a local chart describes a local evolution of the system.

Let us summarize the above situation by looking at the familiar picture in eq. (4.15) by fixing the symplectic structure but instead changing the Hamiltonian. (Note that the magnetic field in the Lorentz force (4.9) does not do any work. So there is no energy flow during the evolution.) At time $t = 0$, the system is described by the free Hamiltonian H_0 but it ends up with the Hamiltonian (4.6) at time $t = 1$. Therefore the dynamics of the system cannot be described with a single time parameter covering the entire period $0 \leq t \leq 1$. We can introduce at most a local time during $\delta t < \epsilon$ on a local patch and smoothly adjust to a neighboring patch. To say, a clock of the particle will tick each time with a different rate since the Hamiltonian of the particle is changing during time evolution.

We have faced a similar situation in the θ-deformation (1.3) as summarized in eq. (4.1). Of course one should avoid a confusion between the dynamical evolution of particle system related to the phase space (1.1) and the dynamical evolution of spacetime geometry related to the NC space (1.3). But we should get an important lesson from Souriau and Sternberg [25] that the Hamiltonian dynamics in the presence of electromagnetic fields can be described by the deformation of symplectic structure of phase space. More precisely, we observed that the emergent geometry is defined by a one-parameter family of diffeomorphisms generated by a smooth vector field X_t satisfying $\iota_{X_t}\omega_t + A = 0$ for the change of a symplectic structure within the same cohomology class from ω to $\omega_t = \omega + t(\omega' - \omega)$ for all $0 \leq t \leq 1$ where $\omega' - \omega = dA$. The vector field X_t is in general not a Hamiltonian flow, so any global time cannot be assigned to the evolution of the symplectic structure ω_t. But, if there is no fluctuation of symplectic structure, i.e., $F = dA = 0$ or $A = -dH$, there can be a globally well-defined Hamiltonian flow. In this case we can define a global time by introducing a unique Hamiltonian such that the time evolution is defined by $df/dt = X_H(f) = \{f, H\}_{\theta=\omega^{-1}}$ everywhere. In particular, when the initial symplectic structure ω is constant (homogeneous), a clock will tick everywhere at the same rate. Note that this situation happens for the constant background (3.1) from which a flat spacetime emerges as we observed in section 3.4. But, if ω is not constant, the time evolution will not be uniform over the space and a clock will tick at the different rate at different places. This is consistent with Einstein gravity since a nonconstant ω corresponds to a curved space in our picture.

We suggest the concept of "Time" in emergent gravity as a contact manifold $(\mathbf{R}\times M, \widetilde{\omega})$ where (M, ω) is a symplectic manifold and $\widetilde{\omega} = \pi_2^*\omega$ is defined by the projection $\pi_2 : \mathbf{R}\times M \to M$, $\pi_2(t, p) = p$. See section 5.1 in [32] for time dependent Hamiltonian systems. A question is then how to recover the (local) Lorentz symmetry in the end. As we pointed out above, if (M, ω) is a canonical symplectic manifold, i.e., $M = \mathbf{R}^{2n}$ and ω=constant, a $(2n+1)$-dimensional Lorentz symmetry will appear from the contact manifold $(\mathbf{R}\times M, \widetilde{\omega})$. (So our $(3 + 1)$-dimensional Lorentzian world needs a more general argument. See the footnote 13.) Once again, the Darboux theorem says that there always exists a local coordinate system where the symplectic structure is of the canonical form. See the table 2. Then it is quite plausible that the local Lorentz symmetry would be recovered in the previous way on a local Darboux chart. Furthermore, the Feynman's argument [26] implies that the Lorentz symmetry is just derived from the symplectic structure on the contact manifold $(\mathbf{R}\times M, \widetilde{\omega})$. For example, one can recover the gauge symmetry along the time direction by defining the Hamiltonian $H = A_0 + H'$ and the time evolution of a spacetime geometry by the Hamilton's equation $D_0 f \equiv df/dt + \{A_0, f\}_{\widetilde{\theta}=\widetilde{\omega}^{-1}} = \{f, H'\}_{\widetilde{\theta}=\widetilde{\omega}^{-1}}$. And then one may interpret the Hamilton's equation as the infinitesimal version of an inner automorphism like eq. (3.17), which was indeed used to define the vector field $D_0(X)$ in eq. (3.30).

Our proposal for the emergent time is based on the fact that a symplectic manifold (M, ω) always admits a Hamiltonian dynamical system on M defined by a Hamiltonian vector field X_H, i.e., $\iota_{X_H}\omega = dH$. The purpose to pose the issue of "Emergent Time" is to initiate and revisit this formidable issue after a deeper understanding of emergent gravity. We refer here some related works for future references: Our proposal is closely related to the picture in [58], where the time is basically defined by a one-parameter group of automorphisms of a von Neumann algebra. Note that the deformation quantization of a Poisson manifold [31] also exhibits a similar automorphism $D(\hbar)$ in eq. (3.62) acting on star-products. section 5.5 in [32] and chapter 21 in [30] (and references therein) provide an exposition on infinite-dimensional Hamiltonian systems, especially, the Hamiltonian formulation of Einstein gravity.

4.2 Matter fields from NC spacetime

Now let us pose our original problem about what matters are in emergent geometry. We will not intend to solve the problem. Instead we will suggest a plausible picture based on the Fermi-surface scenario in [27, 28]. We will return to this problem with more details in the next publication.

Particles are by definition characterized by their positions and momenta besides their intrinsic charges, e.g., spin, isospin and an electric charge. They should be replaced by a matter field in a relativistic quantum theory in order to incorporate pair creations and annihilations. Moreover, in a NC space such as (3.1), the very notion of a point is replaced by a state in the Hilbert space (3.2) and thus the concept of particles (and matter fields too) becomes ambiguous. So a genuine question is what is the most natural notion of a particle or a corresponding matter field in the NC $\star$-algebra (3.3). We suggest it should be a K-theory object in the sense of [27].

Let us briefly summarize the K-theory picture in [27]. Hořava considers nonrelativistic fermions in $(d+1)$-dimensional spacetime having N complex components. Gapless excitations are supported on a $(d-p)$-dimensional Fermi surface Σ in $(\mathbf{k}, \omega)$ space. Consider an inverse exact propagator

$$\mathcal{G}_a{}^{a'} = \delta_a^{a'}(i\omega - \mathbf{k}^2/2m + \mu) + \Pi_a{}^{a'}(\mathbf{k}, \omega) \tag{4.18}$$

where $\Pi_a{}^{a'}(\mathbf{k}, \omega)$ is the exact self-energy and $a, a' = 1, \cdots, N$. Assuming that $\mathcal{G}$ has a zero along a submanifold Σ of dimension $d-p$ in the $(d+1)$-dimensional $(\mathbf{k}, \omega)$ space, the question of stability of the manifold Σ of gapless modes reduces to the classification of the zeros of the matrix $\mathcal{G}$ that cannot be lifted by small perturbations $\Pi_a{}^{a'}$. Consider a sphere $\mathbf{S}^p$ wrapped around Σ in the transverse $p+1$ dimensions in order to classify stable zeros. The matrix $\mathcal{G}$ is nondegenerate along this $\mathbf{S}^p$ and therefore defines a map

$$\mathcal{G} : \mathbf{S}^p \to GL(N, \mathbf{C}) \tag{4.19}$$

from $\mathbf{S}^p$ to the group of nondegenerate complex $N \times N$ matrices. If this map represents a nontrivial class in the pth homotopy group $\pi_p(GL(N, \mathbf{C}))$, the zero along Σ cannot be lifted by a small deformation of the theory. The Fermi surface is then stable under small perturbations, and the corresponding nontrivial element of $\pi_p(GL(N, \mathbf{C}))$ represents the topological invariant responsible for the stability of the Fermi surface. As a premonition, we mention that it is enough to regard the Fermi surface Σ as a (stable) vacuum manifold with a sharp Fermi momentum $\mathbf{p}_F$ where all small excitations are supported, regardless of fermions themselves.

A remarkable point is that there is the so-called stable regime at $N > p/2$ where $\pi_p(GL(N, \mathbf{C}))$ is independent of N. In this stable regime, the homotopy groups of $GL(N, \mathbf{C})$ or $\mathrm{U}(N)$ define a generalized cohomology theory, known as K-theory [59–61]. In K-theory which involves vector bundles and gauge fields, any smooth manifold X is assigned an Abelian group $K(X)$. Aside from a deep relation to D-brane charges and RR fields in string theory [59, 60], the K-theory is also deeply connected with the theory of Dirac operators, index theorem, Riemannian geometry, NC geometry, etc. [41].

Let us look at the action (3.9) recalling that it describes fluctuations around a vacuum, e.g., eq. (3.1). One may identify the map (4.19) with the gauge-Higgs system $(A_\mu, \Phi^a)(z)$ as the maps from $\mathbf{R}^d_C$ to $\mathrm{U}(N \to \infty)$. More precisely, let us identify the $(d-p)$-dimensional Fermi surface Σ with $\mathbf{R}^{2n}_{NC}$ described by eq. (3.1) and the $(p+1)$-dimensional transverse space with $X = \mathbf{R}^d_C$. In this case the Fermi surface Σ is defined by the vacuum (3.1) whose natural energy scale is the Planck energy E_{Pl} as we observed in section 3.4, so the Fermi momentum p_F is basically given by E_{Pl}. The magic of Fermi surface physics is that gapless excitations near the Fermi surface easily forget the possibly huge background energy.

Now we want to consider gapless fluctuations supported on the Fermi surface Σ. The matrix action in eq. (3.9) shows that $\mathbf{R}^d_C$ is not only a hypersurface but also supports a $\mathrm{U}(N \to \infty)$ gauge bundle. This is the reason [60, 61] why $K(X)$ comes into play to classify the topological class of excitations in the $\mathrm{U}(N)$ gauge-Higgs system. As we observed in section 3.4, a generic fluctuation in eq. (3.23) will noticeably deform the background spacetime lattice defined by the Fock space (3.2) and it will generate non-negligible gravitational fields. But our usual concept of particle is that it does not appreciably disturb the ambient gravitational field. This means that the gapless excitation should be a sufficiently localized state in $\mathbf{R}^{2n}_{NC}$. In other words, the state is described by a compact operator in $\mathcal{A}_\theta$, e.g., a Gaussian rapidly vanishing away from $y \sim y_0$ or the matrix elements for a compact operator $\widehat{\Phi} \in \mathcal{A}_\theta$ in the representation (3.5) are mostly vanishing excepts a few elements. A typical example satisfying these properties is NC solitons, e.g., GMS solitons [62].

Since a gauge invariant observable in NC gauge theory is characterized by its momentum variables as we discussed in section 3.2, it will be rather useful to represent the state in momentum space. Another natural property we impose is that it should be stable up to pair creations and annihilations. Therefore it must be generated by the K-theory group of the map (4.19) [59–61], where we will identify the NC $\star$-algebra $\mathcal{A}_\theta$ with $GL(N, \mathbf{C})$ using the relation (3.6). Note that the map (4.19) is contractible to the group of maps from X to $\mathrm{U}(N)$.

With the above requirements in mind, let us find an explicit construction of a topologically non-trivial excitation. It is well-known [61] that this can be done using an elegant construction due to Atiyah, Bott and Shapiro (ABS) [63]. The construction uses the gamma matrices of the transverse rotation group $SO(p,1)$ for $X = \mathbf{R}^d_C$ to construct explicit generators of $\pi_p(U(N))$ where $d = p+1$. Let X be even dimensional and $S_\pm$ be two irreducible spinor representations of $\mathrm{Spin}(d)$ Lorentz group and p_μ ($\mu = 0, 1, \cdots, p$) be the momenta along X, transverse to Σ in $(\mathbf{k}, \omega)$. We define the gamma matrices $\Gamma^\mu : S_+ \to S_-$ of $SO(p,1)$ to satisfy $\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}$. At present we are considering excitations around the constant vacuum (3.1) and so the vacuum geometry is flat. But, if we considered excitations in a nontrivial vacuum such as eq. (3.67), the vacuum manifold might be curved. So the Clifford algebra in this case would be replaced by $\{\Gamma^\mu, \Gamma^\nu\} = 2g^{\mu\nu}$ where the metric $g^{\mu\nu}$ is given by eq. (3.71). Finally we introduce an operator $\mathcal{D} : \mathcal{H} \times S_+ \to \mathcal{H} \times S_-$ [27] such that

$$\mathcal{D} = \Gamma^\mu p_\mu + \cdots \tag{4.20}$$

which is regarded as a linear operator acting on a Hilbert space $\mathcal{H}$, possibly much smaller than the Fock space (3.2), as well as the spinor vector space $S_\pm$.

The ABS construction implies [27, 28] that the Dirac operator (4.20) is a generator of $\pi_p(U(N))$ as a nontrivial topology in momentum space $(\mathbf{k}, \omega)$ where the low lying excitations in eq. (4.19) near the Fermi surface Σ carry K-theory charges and so they are stable. Such modes are described by coarse-grained fermions $\chi^A(\omega, \mathbf{p}, \theta)$ with θ denoting collective coordinates on Σ and $\mathbf{p}$ being the spatial momenta normal to Σ [27]. The ABS construction determines the range $\widetilde{N}$ of the index A carried by the coarse-grained fermions χ^A to be $\widetilde{N} = 2^{[p/2]} n \leq N$ complex components. The precise form of the fermion χ^A depends on its K-theory charge whose explicit representation on $\mathcal{H} \times S_\pm$ will be determined later. And we will apply the Feynman's approach [26] to see what the multiplicity n means. For a moment, we put $n = 1$. At low energies, the dispersion relation of the fermion χ^A near the Fermi surface is given by the relativistic Dirac equation

$$i\Gamma^\mu \partial_\mu \chi + \cdots = 0 \tag{4.21}$$

with possible higher order corrections in higher energies. Thus we get a spinor of the Lorentz group $SO(p,1)$ from the ABS construction as a topological solution in momentum space. For example, in four dimensions, i.e., $p = 3$, χ^A has two complex components and so it describes a chiral Weyl fermion.

Although the emergence of $(p+1)$-dimensional spinors is just a consequence due to the fact that the ABS construction uses the Clifford algebra to construct explicit generators of $\pi_p(U(N))$, it is mysterious and difficult to understand its physical origin. But we believe that the fermionic nature of the excitation χ is originated from some unknown Planck scale physics. For example, if the Dirac operator (4.20) is coming from GMS solitons [62] in $\mathbf{R}^{2n}_{NC}$, the GMS solitons correspond to eigenvalues of $N \times N$ matrices in eq. (3.6). As is well known from $c = 1$ matrix models, the eigenvalues behave like fermions, although it is the (1+1)-dimensional sense, after integrating out off-diagonal interactions. Another evidence is the stringy exclusion principle [64] that the AdS/CFT correspondence puts a limit on the number of single particle states propagating on the compact spherical component of the $AdS_p \times \mathbf{S}^q$ geometry which corresponds to the upper bound on U(1) charged chiral primaries on the compact space $\mathbf{S}^q$.

It should be important to clearly understand the origin of the fermionic nature of particles arising from the vacuum (3.1). The crux seems to be the mysterious connection between the Clifford modules and K-theories [63]. Another related problem is that we didn't yet understand the dynamical origin of the particle symplectic structure (4.2). Is it similarly possible to get some insight about the particle mass and dark matters from the dynamical origin of the symplectic structure (4.2) as we did in section 3.4 for the dark energy ? If the vacuum (3.1) acts as a Fermi surface for quarks and leptons, is it a symptom that the local electroweak symmetry can be broken dynamically without Higgs ?

Now let us address the problem how to determine the multiplicity n of the coarse-grained fermions $\chi^{\alpha a}$ where we decomposed the index $A = (\alpha a)$ with α the spinor index of the SO(d) Lorentz group and $a = 1, \cdots, n$ an internal index of an n-dimensional representation of some compact symmetry G. One may address this problem by considering the quantum particle dynamics on $X \times \Sigma$ and repeating the Feynman's question. To be specific, we restrict (collective) coordinates of Σ, denoted as Q^I ($I = 1, \cdots, n^2 - 1$), to Lie algebra variables such as the particle isospins or colors. So the commutation relations we consider are

$$[Q^I, Q^J] = if^{IJK} Q^K, \tag{4.22}$$

$$[q^i, Q^I] = 0 \tag{4.23}$$

together with the commutation relations (4.17) determined by the symplectic structure (4.2) on $T^*\mathbf{R}^p$.

Then the question is: What is the most general form of forces consistent with the commutation relations (4.17), (4.22) and (4.23) ? It was already answered in [56] that the answer is just the non-Abelian version of the Lorentz force law (4.16) with an additional set of equations coming from the condition that the commutation relation (4.23) should be preserved during time evolution, i.e., $\frac{d}{dt}[q^i, Q^I] = 0$. This condition can be solved by the so-called Wong's equations

$$\dot{Q}^I + f^{IJK} A_i^J Q^K \dot{q}_i = 0. \tag{4.24}$$

The Wong's equations just say that the internal charge Q^I is parallel-transported along the trajectory of the particle under the influence of the non-Abelian gauge field A_i^J.

Therefore the quantum particle dynamics on $X \times \Sigma$ naturally requires to introduce non-Abelian gauge fields in the representation of the Lie algebra (4.22). And the dynamics of the particle carrying an internal charge in Σ should be defined by a symplectic structure on $T^*X \times \Sigma$. But note that we have a natural symplectic structure on Σ defined by eq. (3.1). Also note that we have only U(1) gauge fields on $X \times \Sigma$ in eq. (3.7). So the problem is how to get the Lie algebra generators in eq. (4.22) from the space $\Sigma = \mathbf{R}^{2n}_{NC}$ and how to get the non-Abelian gauge fields $A^I_\mu(z)$ on X from the U(1) gauge fields on $X \times \Sigma$ where $z^\mu = (t, q^i)$.

The problem is solved by noting that the n-dimensional harmonic oscillator in quantum mechanics can realize SU(n) symmetries (see the chapter 14 in [65]). The generators of the SU(n) symmetry on the Fock space (3.2) are given by

$$Q^I = a_i^\dagger T^I_{ik} a_k \tag{4.25}$$

where the creation and annihilation operators are given by eq. (3.1) and T^I's are constant $n \times n$ matrices satisfying $[T^I, T^J] = if^{IJK}T^K$ with the same structure constants as eq. (4.22). It is easy to check that the Q^I's satisfy the SU(n) Lie algebra (4.22). We introduce the number operator $Q^0 \equiv a_i^\dagger a_i$ and identify with a U(1) generator. The operator $\mathfrak{C} = \sum_I Q^I Q^I$ is the quadratic Casimir operator of the SU(n) Lie algebra and commutes with all Q^I's. Thus one may identify $\mathfrak{C}$ with an additional U(1) generator.

Let $\rho(V)$ be a representation of the Lie algebra (4.22) in a vector space V. We take an n-dimensional representation in $V = \mathbf{C}^n$ or precisely $V = L^2(\mathbf{C}^n)$, a square integrable Hilbert space. Now we expand the U(1) gauge field $\widehat{A}_M(z,y)$ in eq. (3.7) in terms of the SU(n) basis (4.25)

$$\begin{aligned} \widehat{A}_M(z,y) &= \sum_{n=0}^{\infty} \sum_{I_i \in \rho(V)} A_M^{I_1 \cdots I_n}(z,\rho,\lambda_n)\, Q^{I_1} \cdots Q^{I_n} \\ &= A_M(z) + A_M^I(z,\rho,\lambda_1)\, Q^I + A_M^{IJ}(z,\rho,\lambda_2)\, Q^I Q^J + \cdots \end{aligned} \tag{4.26}$$

where ρ and λ_n are eigenvalues of Q^0 and $\mathfrak{C}$, respectively, in the representation $\rho(V)$. The expansion (4.26) is formal but it is assumed that each term in eq. (4.26) belongs to the irreducible representation of $\rho(V)$. Thus we get SU(n) gauge fields A^I_μ as well as adjoint scalar fields A^I_a in addition to U(1) gauge fields $A_M(z)$ as low lying excitations.

Note that the coarse-grained fermion χ in eq. (4.21) behaves as a relativistic particle in the spacetime $X = \mathbf{R}^d_C$ and a stable excitation as long as the Fermi surface Σ is topologically stable. In addition to these fermionic excitations, there will also be bosonic excitations arising from changing the position in X of the surface Σ or deformations of the surface Σ itself. But the latter effect (as gravitational fields in Σ) will be very small and so can be ignored since we are interested in the low energy behavior of the Fermi surface Σ. Then the gauge fields in eq. (4.26) represent collective modes for the change of the position in $X = \mathbf{R}^d_C$ of the surface Σ [28]. They can be regarded as collective dynamical fields in the vicinity of the Fermi surface Σ acting on the fermions in eq. (4.21).

Therefore we regard the Dirac operator (4.20) as an operator $\mathcal{D}: \mathcal{H} \times S_+ \to \mathcal{H} \times S_-$ where $\mathcal{H} = L^2(\mathbf{C}^n)$ and introduce a minimal coupling with the U(1) and SU(n) gauge fields in eq. (4.26) by the replacement $p_\mu \to p_\mu - eA_\mu - A^I_\mu Q^I$. Then the Dirac equation (4.21) becomes

$$i\Gamma^\mu(\partial_\mu - ieA_\mu - iA^I_\mu Q^I)\chi + \cdots = 0. \tag{4.27}$$

Here we see that the coarse-grained fermion χ in the homotopy class $\pi_p(\mathrm{U}(N))$ is in the fundamental representation of SU(n). So we identify the multiplicity n in the ABS construction (4.21) with the number of colors. Unfortunately the role of the adjoint scalar fields in eq. (4.26) is not obvious from the Feynman's approach.

The most interesting case in eq. (3.9) is that $p = 3$ and $n = 3$, that is, 10-dimensional NC U(1) gauge theory on $\mathbf{R}^4_C \times \mathbf{R}^6_{NC}$. In this case eq. (4.27) is the 4-dimensional Dirac equation where χ is a quark, an SU(3) multiplet of chiral Weyl fermions, coupling with gluons $A^I_\mu(z)$, SU(3) gauge fields for the color charge Q^I, as well as photons $A_\mu(z)$, U(1) gauge fields for the electric charge e. One may consider a similar ABS construction in the vector space $V = \mathbf{C}^2 \times \mathbf{C}$, i.e., by breaking the SU(3) symmetry to SU(2) $\times$ U(1), to get SU(2) gauge fields and chiral Weyl fermions. In this case Q^I $(I = 1, 2, 3)$ in eq. (4.25) are the famous Schwinger representation of SU(2) Lie algebra.

5 Musing on noncommutative spacetime

It is a well-accepted consensus that at very short distances, e.g., the Planck scale L_P, the spacetime is no longer commutating due to large quantum effects and a NC geometry will play a role at short distances. In addition, the spacetime geometry at the Planck scale is not fixed but violently fluctuating, as represented as spacetime foams. Therefore the NC geometry arising at very short distances has to be intimately related to quantum gravity. The Moyal space (1.3) is the simplest and the most natural example of NC spacetime. Thus it should be expected that the physical laws defined in the NC spacetime (1.3), for instance, a NC field theory, essentially refer to a theory of (quantum) gravity. This is the reason why the θ-deformation in the table 1 must be radical as much as the $\hbar$-deformation.

Unfortunately, the NC field theory has not been explored as a theory of gravity so far. It has been studied as a theory of particles within the conventional framework of quantum field theory. But we have to recognize that the NC field theory is a quantum field theory defined in a highly nontrivial vacuum (3.1). It should be different from usual quantum field theories defined in a trivial vacuum. So we should be careful to correctly identify order parameters for fluctuations around the vacuum (3.1). We may have a wrong choice of the order parameter if we naively regard the NC field theory as a theory of particles only. As an illustrating example, in order to describe the superconductivity at $T \lesssim T_c$, it is important to consider an effect of the background lattice and phonon exchange with electrons. The interaction of electrons with the background lattice is resulted in a new order parameter, the so-called Cooper pairs, and a new attractive force between them. We know that it is impossible to have a bound state of two electrons, the Cooper pair, in a trivial vacuum, i.e., without the background lattice. Thus the superconductivity is an emergent phenomenon from electrons moving in a nontrivial background lattice.

We observed that the vacuum (3.1) endows the spacetime with a symplectic structure whose surprising consequences, we think, have been considerably explored in this paper. For example, it brings to the correspondence (3.6) implying the large N duality or the gauge/gravity duality. These features do not arise in ordinary quantum field theories. So it would be desirable to seriously contemplate about the theoretical structure of NC field theories from the spacetime point of view.

JHEP05(2009)012

5.1 Graviton as a Cooper pair

Graviton is a spin-2 particle. Therefore the emergent gravity, if the picture is true, should come from a composite of two spin-1 gauge bosons, not from gauge fields themselves.[22] Unfortunately, there is no rigorous proof that the bound state of two spin-1 gauge bosons does exist in NC spacetime. But an interesting point is that NC spacetime is more preferable to the formation of bound states compared to commutative spacetime. See, for example, [66]. Salient examples are GMS solitons [62] and NC U(1) instantons [67], which are not allowed in a commutative spacetime. Furthermore there are many logical evidences that it will be true, especially inferred from the matrix formulation of NC gauge theory as we briefly discuss below.

For definiteness, let us consider the case with $d = 4$ and $n = 3$ for the action (3.9), that is, 10-dimensional NC U(1) gauge theory on $\mathbf{R}^4_C \times \mathbf{R}^6_{NC}$. The matrix representation in the action (3.9) is precisely equal to the bosonic part of 4-dimensional $\mathcal{N} = 4$ supersymmetric U(N) Yang-Mills theory which is known to be equivalent to the type IIB string theory on $AdS_5 \times \mathbf{S}^5$ space [24]. Therefore the 10-dimensional gravity emergent from NC gauge theory will essentially be the same as the one in the AdS/CFT duality. The bulk graviton $g_{\mu\nu}(z, \rho)$ in the AdS/CFT duality, whose asymptotics at $\rho = 0$ is given by the metric (3.89), is defined by the coupling to the energy-momentum tensor $T_{\mu\nu}(z)$ in the U(N) gauge theory. The energy-momentum tensor $T_{\mu\nu}(z)$ is a spin-2 composite operator in the gauge theory rather than a fundamental field. This means that the bulk graviton is holographically defined as a bound state of two spin-1 gauge bosons. Schematically, we have the following relation

$$(1 \otimes 1)_S \ \rightleftarrows\ 2 \oplus 0 \quad \text{or} \quad \subset \otimes \supset \rightleftarrows \bigcirc. \tag{5.1}$$

Indeed the core relation (5.1) has underlain the unification theories since Kaluza and Klein. In early days people have tried the scheme ($\leftarrow$) under the name of the Kaluza-Klein theory. A basic idea in the Kaluza-Klein theory (including string theory) is to construct spin-1 gauge fields plus gravity in lower dimensions from spin-2 gravitons in higher dimensions. An underlying view in this program is that a "fundamental" theory exists as a theory of gravity in higher dimensions and a lower dimensional theory of spin-1 gauge fields is derived from the higher dimensional gravitational theory. Though it is mathematically beautiful and elegant, it seems to be physically unnatural if the higher spin theory should be regarded as a more fundamental theory.

After the discovery of D-branes in string theory, people have realized that the scheme ($\rightarrow$) is also possible, which is now known as the open-closed string duality or the gauge/gravity duality. But the scheme ($\rightarrow$) comes into the world in a delicate way since there is a general no-go theorem known as the Weinberg-Witten theorem [68, 69], stating that an interacting graviton cannot emerge from an ordinary quantum field theory in the same spacetime. One has to notice, however, that Weinberg and Witten introduced two basic assumptions to prove this theorem. The first hidden assumption is that gravitons and gauge fields live in the same spacetime. The second assumption is the existence of a Lorentz-covariant stress-energy tensor. The AdS/CFT duality [24] realizes the emergent

[22] We thank Piljin Yi for raising this critical issue.

gravity by relaxing the first assumption in the way that gravitons live in a higher dimensional spacetime than gauge fields. As we observed in section 3.4, the NC field theory is even more radical in the sense that the Lorentz symmetry is not a fundamental symmetry of the theory but emergent from the vacuum algebra (3.1) defined by a uniform configuration of NC gauge fields.

Another ingredient supporting the existence of the spin-2 bound states is that the vacuum (3.1) in NC gauge theory signifies the spontaneous symmetry breaking of the Λ-symmetry (2.11) [3]. If one considers a small fluctuation around the vacuum (3.1) parameterized by eq. (2.32), the spacetime metric given by eq. (3.42) looks like

$$g_{MN} = \eta_{MN} + h_{MN} \tag{5.2}$$

where $\eta_{MN} = \langle g_{MN} \rangle$ is the flat metric determined by the uniform condensation of gauge fields in the vacuum. As a fluctuating (quantum) field, the existence of the vacuum expectation value in the metric $\langle g_{MN} \rangle = \eta_{MN}$ also implies some sort of spontaneous symmetry breaking as Zee anticipated in [40] (see the footnote 8). We see here that they indeed have the same origin. If one look at the table 2, one can see a common property that both a Riemannian metric g and a symplectic structure ω should be nondegenerate, i.e., nowhere vanishing on M. In the context of physics where g and ω are regarded as a field, the nondegeneracy means a nonvanishing vacuum expectation value. We refer to [3] more discussions about the spontaneous symmetry breaking.

Instead we will discuss an interesting similarity between the BCS superconductivity [29] and the emergent gravity to get some insight into the much more complicated spontaneous symmetry breaking for the Λ-symmetry (2.11). A superconductor of any kind is nothing more or less than a material in which the $G = \mathrm{U}(1)$ gauge symmetry is spontaneously broken to $H = \mathbf{Z}_2$ which is the 180^o phase rotation preserved by Cooper pairs [70]. The spontaneous breakdown of electromagnetic gauge invariance arises because of attractive forces between electrons via lattice vibrations near the Fermi surface. In consequence of this spontaneous symmetry breaking, products of any even number of electron fields have nonvanishing expectation values in a superconductor, captured by the relation $\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$. As we mentioned above, the emergent gravity reveals a similar pattern of spontaneous symmetry breaking though much more complicated where the Λ-symmetry (2.11), or equivalently $G = \mathrm{Diff(M)}$, is spontaneously broken to the symplectomorphism (2.23), or equivalently $H = \mathrm{U}(1)_{NC}$ gauge symmetry. The spontaneous breakdown of the Λ-symmetry or $G = \mathrm{Diff(M)}$ is induced by the condensate (3.1) of gauge fields in a vacuum and conceivably the vacuum (3.1) can act as a Fermi surface for low energy excitations, as we discussed in section 4.2.

Then we may find a crude but inciting analogy between the BCS superconductivity and the emergent gravity:

The Landau-Ginzburg theory is a phenomenological theory of superconductivity where the free energy of a superconductor near $T \approx T_c$ can be expressed in terms of a complex order parameter, describing Cooper pairs [70]. Of course this situation is analogous to the emergent gravity in the sense that Einstein gravity as a macroscopic description of NC gauge fields is manifest only at the commutative limit, i.e., $|\theta| \to 0$. Although we should

Theory	Superconductivity	Emergent gravity
Microscopic degree of freedom	electron	gauge field
Order parameter	Cooper pair	graviton
G	U(1)	Diff(M)
H	$\mathbf{Z}_2$	$\mathrm{U}(1)_{NC}$
Control parameter	$\frac{T_c}{T}-1$	θ^{ab}
Macroscopic description	Laudau-Ginzburg	Einstein gravity
Microscopic description	BCS	gauge theory

Table 3. Superconductivity vs. Emergent gravity.

JHEP05(2009)012

be cautious to employ the analogy in the table 3, it may be worthwhile to remark that the flux tubes or Abrikosov vortices in type II superconductors, realized as a soliton solution in the Landau-Ginzburg theory, seem to be a counterpart of black holes in the emergent gravity. We think the table 3 could serve as a guidepost more than a plain analogy to understand a detailed structure of emergent gravity.

5.2 Fallacies on noncommutative spacetime

As was remarked before, a NC spacetime arises as a result of large quantum fluctuations at very short distances. So the conventional spacetime picture gained from a classical and weak gravity regime will not be naively extrapolated to the Planck scale. Indeed we perceived that a NC geometry reveals a novel, radically different picture about the origin of spacetime.

But the orthodox approach so far has regarded the NC spacetime described by eq. (3.1), for instance, as an additional background condensed on an already existing spacetime. For example, field theories defined on the NC spacetime have been studied from the conventional point of view based on the traditional spacetime picture. Then the NC field theory is realized with unpleasant features, breaking the Lorentz symmetry and locality which are two fundamental principles underlying quantum field theories. A particle in local quantum field theories is defined as a state in an irreducible representation of the Poincaré symmetry and internal symmetries. This concept of the particle becomes ambiguous in the NC field theory due to not only the lack of the Lorentz symmetry but also the non-Abelian nature of spacetime. Furthermore the nonlocality in NC field theories appears as a perplexing UV/IR mixing in nonplanar Feynman diagrams in perturbative dynamics [55]. This would appear to spoil the renormalizability of these theories [1].

Therefore the NC field theory is not an eligible generalization of quantum field theory framework as a theory of particles. However, these unpleasant aspects of the NC field theory turn into a welcome property or turn out to be a fallacy whenever one realizes it as a theory of gravity. We believe that the nonperturbative dynamics of gravity is intrinsically nonlocal. A prominent evidence is coming from the holographic principle [71] which states that physical degrees of freedom in gravitational theories reside on a lower dimensional screen where gauge fields live. The AdS/CFT duality [24] is a thoroughly tested example of the holographic principle. Recently it was shown in [18, 19] that the UV/IR mixing in

NC gauge theories can be interpreted as a manifestation of gravitational nonlocality in the context of emergent gravity. This elegant shift of wing signifies an internal consistency of emergent gravity.

The basic idea of emergent gravity is to view the gravity as a collective phenomenon of gauge fields. According to Einstein, the gravity is nothing but the dynamics of spacetime geometry. This perspective implies that there is no prescribed notion of spacetime. The spacetime must also be emergent from or defined by gauge fields if the picture is anyway correct. We observed in section 3.4 that the emergent gravity reveals a novel and consistent picture about the dynamical origin of spacetime. The most remarkable angle is the dynamical origin of flat spacetime, which is absent in Einstein gravity. It turned out that the Lorentz symmetry as well as the flat spacetime is not *a priori* given in the beginning but emergent from or defined by the uniform condensation (3.1) of gauge fields. In the prospect, the Lorentz symmetry is not broken by the background (3.1) but rather emergent at the cost of huge energy condensation in the vacuum. Thus the emergent gravity also comes to the rescue of the Lorentz symmetry breaking in NC field theories.

But we want to point out an intriguing potential relation between the dark energy (3.94) and a possible tiny violation of the Lorentz symmetry. We observed that the energy density (3.94) is due to the cosmological vacuum fluctuation around the flat spacetime and does generate an observable effect of spacetime structure, e.g., an expansion of universe. Furthermore, since the tiny energy (3.94) represents a deviation from the flat spacetime over the cosmological scale, it may have another observable effect of spacetime structure; a very tiny violation of the Lorentz symmetry. Amusingly, the dark energy scale $\sim 2 \times 10^{-3} eV$ given by (3.93) is of the same order of magnitude as the neutrino mass. This interesting numerical coincidence may imply some profound relation between the neutrino mass and the tiny violation of the Lorentz symmetry [72].

6 Discussion

> *Mathematicians do not study objects, but relations between objects. Thus, they are free to replace some objects by others so long as the relations remain unchanged. Content to them is irrelevant: they are interested in form only.*
>
> – Henri Poincaré (1854-1912)

Recent developments in string and M theories, especially, after the discovery of D-branes, have constantly revealed that string and M theories are not very different from quantum field theories. Indeed a destination of nonperturbative formulations of string and M theories has often been quantum field theories again. For instance, the AdS/CFT duality and the matrix models in string and M theories are only a few salient examples. It seems to insinuate a message that quantum field theories already contain 'quantum gravity' in some level. At least we have to contemplate our credulous belief that the string and M theories should be superordinate to quantum field theories. Certainly we are missing the first (dynamical) principle to derive the quantum gravity from quantum field theories.

Throughout the paper, we have emphasized that quantum field theories in NC spacetime are radically different from their commutative counterparts and they should be regarded as a theory of gravity rather than a theory of particles. So the important message we want to draw is that the θ-deformation in the table 1 should be seriously considered as a foundation for quantum gravity. In other words, the first principle would be the geometrization of gauge fields based on the symplectic and NC geometry. It my be possible that the NC geometry also underlies the fundamentals of string theory.

In this paper, we have mostly focussed on the commutative limit, $\theta \to 0$, where the Einstein gravity manifests itself as a macroscopic spacetime geometry of NC $\star$-algebra defined by gauge fields in NC spacetime. That is, Einstein gravity is just a low energy effective theory of NC gauge fields or large N matrices. So we naturally wonder what happens in a deep NC space. An obvious guess is that a usual concept of spacetime based on a smooth geometry will be doomed. Instead an operator algebra, e.g., $\star$-algebra defined by NC gauge fields, will define a relational fabric between NC gauge fields, whose prototype at macroscopic world emerges as a smooth spacetime geometry. In a deep NC space, an algebra between objects is more fundamental. A geometry is a secondary concept defined by the algebra. Indeed the motto in emergent gravity is that an algebra defines a geometry. In this scheme, one has to specify an underlying algebra to talk about a corresponding geometry. So the Poincaré's declaration above may also refer to physicists who are studying quantum gravity.

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A A proof of the equivalence between self-dual NC electromagnetism and self-dual Einstein gravity

Here we present a self-contained and friendly proof of the equivalence between self-dual NC electromagnetism and self-dual Einstein gravity [12]. Our proof here closely follows the result in [73] applying our observation (3.23), of course, decisive for the equivalence, that NC gauge fields can be mapped to (generalized) vector fields through the inner automorphism (3.17). The self-dual case here will be a useful guide for deriving the general equivalence between NC U(1) gauge theories and Einstein gravity presented in appendix B.

We introduce at each spacetime point in M a local frame of reference in the form of 4 linearly independent vectors (vierbeins or tetrads) $E_A = E_A^M \partial_M \in TM$ which are chosen to be orthonormal, i.e., $E_A \cdot E_B = \delta_{AB}$. The basis $\{E_A\}$ determines a dual basis $E^A = E^A_M dX^M \in T^*M$ by

$$\langle E^A, E_B \rangle = \delta^A_B. \tag{A.1}$$

The above pairing leads to the relation $E^A_M E^M_B = \delta^A_B$. The metric is the most basic invariant defined by the vectors in TM or T^*M,

$$\left(\frac{\partial}{\partial s}\right)^2 = \delta^{AB} E_A \otimes E_B = \delta^{AB} E^M_A E^N_B \, \partial_M \otimes \partial_N \equiv g^{MN}(X) \, \partial_M \otimes \partial_N \tag{A.2}$$

or

$$ds^2 = \delta_{AB} E^A \otimes E^B = \delta_{AB} E^A_M E^B_N \, dX^M \otimes dX^N \equiv g_{MN}(X) \, dX^M \otimes dX^N. \tag{A.3}$$

Under local frame rotations in SO(4) the vectors transform according to

$$\begin{aligned} E_A(X) \to E'_A(X) &= E_B(X) \Lambda^B{}_A(X), \\ E^A(X) \to E'^A(X) &= \Lambda^A{}_B(X) E^B(X) \end{aligned} \tag{A.4}$$

where $\Lambda^A{}_B(X) \in$ SO(4). The spin connections $\omega_M(X)$ constitute gauge fields with respect to the local SO(4) rotations

$$\omega_M \to \Lambda \omega_M \Lambda^{-1} + \Lambda \partial_M \Lambda^{-1} \tag{A.5}$$

and the covariant derivative is defined by

$$\begin{aligned} \mathcal{D}_M E_A &= \partial_M E_A - \omega_M{}^B{}_A E_B, \\ \mathcal{D}_M E^A &= \partial_M E^A + \omega_M{}^A{}_B E^B. \end{aligned} \tag{A.6}$$

The connection one-form $\omega^A{}_B = \omega_M{}^A{}_B dX^M$ satisfies the Cartan's structure equations [30],

$$T_{MN}{}^A = \partial_M E^A_N - \partial_N E^A_M + \omega_M{}^A{}_B E^B_N - \omega_N{}^A{}_B E^B_M, \tag{A.7}$$

$$R_{MN}{}^A{}_B = \partial_M \omega_N{}^A{}_B - \partial_N \omega_M{}^A{}_B + \omega_M{}^A{}_C \omega_N{}^C{}_B - \omega_N{}^A{}_C \omega_M{}^C{}_B, \tag{A.8}$$

where we introduced the torsion two-form $T^A = \frac{1}{2} T_{MN}{}^A dX^M \wedge dX^N$ and the curvature two-form $R^A{}_B = \frac{1}{2} R_{MN}{}^A{}_B dX^M \wedge dX^N$. Now we impose the torsion free condition, $T_{MN}{}^A = \mathcal{D}_M E^A_N - \mathcal{D}_N E^A_M = 0$, to recover the standard content of general relativity, which eliminates ω_M as an independent variable, i.e.,

$$\begin{aligned} \omega_{MBC} &= \frac{1}{2} E^A_M (f_{ABC} - f_{BCA} + f_{CAB}) \\ &= -\omega_{MCB}, \end{aligned} \tag{A.9}$$

where

$$f_{ABC} = E^M_A E^N_B (\partial_M E_{NC} - \partial_N E_{MC}). \tag{A.10}$$

Note that $f_{AB}{}^C$ are the structure functions of the vectors $E_A \in TM$ defined by

$$[E_A, E_B] = -f_{AB}{}^C E_C. \tag{A.11}$$

Here raising and lowering the indices $A, B, \cdots$ are insignificant with Euclidean signature but we have kept track of the position of the indices for another use with Lorentzian signature.

Since the spin connection ω_{MAB} and the curvature tensor R_{MNAB} are antisymmetric on the AB index pair, one can decompose them into a self-dual part and an anti-self-dual part as follows

$$\begin{aligned} \omega_{MAB} &= \omega_M^{(+)a} \eta^a_{AB} + \omega_M^{(-)a} \bar{\eta}^a_{AB}, && \text{(A.12)} \\ R_{MNAB} &= F^{(+)a}_{MN} \eta^a_{AB} + F^{(-)a}_{MN} \bar{\eta}^a_{AB} && \text{(A.13)} \end{aligned}$$

where the 4×4 matrices η^a_{AB} and $\bar{\eta}^a_{AB}$ for $a = 1, 2, 3$ are 't Hooft symbols defined by

$$\begin{aligned} \bar{\eta}^a_{ij} &= \eta^a_{ij} = \varepsilon_{aij}, \qquad i, j \in \{1, 2, 3\}, \\ \bar{\eta}^a_{4i} &= \eta^a_{i4} = \delta_{ai}. && \text{(A.14)} \end{aligned}$$

We list some identities of the 't Hooft tensors

$$\begin{aligned} \eta^{(\pm)a}_{AB} &= \pm\frac{1}{2}\varepsilon_{AB}{}^{CD}\eta^{(\pm)a}_{CD}, && \text{(A.15)} \\ \eta^{(\pm)a}_{AB}\eta^{(\pm)a}_{CD} &= \delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC} \pm \varepsilon_{ABCD}, && \text{(A.16)} \\ \varepsilon_{ABCD}\eta^{(\pm)a}_{DE} &= \mp(\delta_{EC}\eta^{(\pm)a}_{AB} + \delta_{EA}\eta^{(\pm)a}_{BC} - \delta_{EB}\eta^{(\pm)a}_{AC}), && \text{(A.17)} \\ \eta^{(\pm)a}_{AB}\eta^{(\mp)b}_{AB} &= 0, && \text{(A.18)} \\ \eta^{(\pm)a}_{AC}\eta^{(\pm)b}_{BC} &= \delta^{ab}\delta_{AB} + \varepsilon^{abc}\eta^{(\pm)c}_{AB}, && \text{(A.19)} \\ \eta^{(\pm)a}_{AC}\eta^{(\mp)b}_{BC} &= \eta^{(\mp)b}_{AC}\eta^{(\pm)a}_{BC} && \text{(A.20)} \end{aligned}$$

where $\eta^{(+)a}_{AB} = \eta^a_{AB}$ and $\eta^{(-)a}_{AB} = \bar{\eta}^a_{AB}$. (Since the above 't Hooft tensors are defined in Euclidean $\mathbf{R^4}$ where the flat metric δ_{AB} is used, we don't concern about raising and lowering the indices.)

Using the identities (A.19) and (A.20), it is easy to see that the (anti-)self-dual curvature in eq. (A.13) is purely determined by the (anti-)self-dual spin connection without any mixing, i.e.,

$$F^{(\pm)a}_{MN} = \partial_M \omega^{(\pm)a}_N - \partial_N \omega^{(\pm)a}_M - 2\varepsilon^{abc}\omega^{(\pm)b}_M \omega^{(\pm)c}_N. \quad \text{(A.21)}$$

Of course all these separations are due to the fact, $SO(4) = SU(2)_L \times SU(2)_R$, stating that any $SO(4)$ rotations can be decomposed into self-dual and anti-self-dual rotations. Since ε^{abc} is the structure constant of $SU(2)$ Lie algebra, i.e., $[\tau^a, \tau^b] = 2i\varepsilon^{abc}\tau^c$ where τ^a's are the Pauli matrices, one may identify $\omega^{(\pm)a}_M$ with $SU(2)_{L,R}$ gauge fields and $F^{(\pm)a}_{MN}$ with their field strengths.

In consequence we have arrived at the following important property. If the spin connection is self-dual, i.e., $\omega^{(-)a}_M = 0$, the curvature tensor is also self-dual, i.e., $F^{(-)a}_{MN} = 0$. Conversely, if the curvature is self-dual, i.e., $F^{(-)a}_{MN} = 0$, one can always choose a self-dual spin connection by a suitable gauge choice since $F^{(-)a}_{MN} = 0$ requires that $\omega^{(-)a}_M$ is a pure gauge. Therefore, in this self-dual gauge, the problem of finding a self-dual solution to the Einstein equation [74]

$$R_{MNAB} = \pm\frac{1}{2}\varepsilon_{AB}{}^{CD}R_{MNCD} \quad \text{(A.22)}$$

is equivalent to one of finding self-dual spin connections

$$\omega_{EAB} = \pm \frac{1}{2} \varepsilon_{AB}{}^{CD} \omega_{ECD} \tag{A.23}$$

where $\omega_{CAB} = E_C^M \omega_{MAB}$. Note that a metric satisfying the self-duality equation (A.22), known as the gravitational instanton, is necessarily Ricci-flat because $R_{MBA}{}^B = \pm \frac{1}{6} \varepsilon_A{}^{BCD} R_{M[BCD]} = 0$. The gravitational instantons defined by eq. (A.22) are then obtained by solving the first-order differential equations given by eq. (A.23).

Now contracting $\varepsilon_F{}^{EAB}$ on both sides of eq. (A.23) leads to the relation

$$\omega_{[ABC]} = \mp \varepsilon_{ABC}{}^D \phi_D \tag{A.24}$$

where $\phi_D = \omega_{ED}{}^E$ and $\omega_{[ABC]} = \omega_{ABC} + \omega_{BCA} + \omega_{CAB}$. From eqs.(A.9) and (A.10) together with eq. (A.24), we get

$$\begin{aligned} f_{ABC} &= \omega_{ABC} - \omega_{BAC} = -\omega_{ACB} - \omega_{BAC} - \omega_{CBA} + \omega_{CBA} \\ &= \pm \varepsilon_{ACB}{}^D \phi_D - \omega_{CAB} \end{aligned} \tag{A.25}$$

and so

$$-\omega_{CAB} = f_{ABC} \pm \varepsilon_{ABC}{}^D \phi_D. \tag{A.26}$$

The self-duality equation (A.23) now can be understood as that of the right-hand side of eq. (A.26) with respect to the AB index pair. In addition the combination $\phi_{[A}\delta_{B]C} \mp \varepsilon_{ABC}{}^D \phi_D$ also satisfies the same type of the self-duality equation with respect to the AB index pair. So we see that the combination $f_{AB}{}^C + \phi_{[A}\delta_{B]}^C$ also satisfies the same self-duality equation

$$f_{AB}{}^E + \phi_{[A}\delta_{B]}^E = \pm \frac{1}{2} \varepsilon_{AB}{}^{CD} \Big(f_{CD}{}^E + \phi_{[C}\delta_{D]}^E \Big). \tag{A.27}$$

Let us introduce a volume form $\mathfrak{v} = \lambda^{-1} \mathfrak{v}_g$ for some function λ where

$$\mathfrak{v}_g = E^1 \wedge E^2 \wedge E^3 \wedge E^4. \tag{A.28}$$

Suppose that E_A's preserve the volume form $\mathfrak{v}$, i.e., $\mathcal{L}_{E_A} \mathfrak{v} = 0$ which is always possible, as rigorously proved in [75], by considering an SO(4) rotation (A.4) of basis vectors and choosing the function λ properly [23]. This leads to the relation $\mathcal{L}_{E_A} \mathfrak{v} = (\nabla \cdot E_A - E_A \log \lambda) \mathfrak{v} = 0$. Since $\nabla \cdot E_A = -\omega_{BA}{}^B = -\phi_A$, we get the identity $\phi_A = -E_A \log \lambda$ for the volume form $\mathfrak{v}$. Define $D_A \equiv \lambda E_A \in TM$. Then we have

$$\begin{aligned} [D_A, D_B] &= \lambda \Big(-f_{AB}{}^C + E_A \log \lambda \delta_B^C - E_B \log \lambda \delta_A^C \Big) D_C \\ &= -\lambda \Big(f_{AB}{}^C + \phi_{[A}\delta_{B]}^C \Big) D_C. \end{aligned} \tag{A.29}$$

Finally we get from eq. (A.27) the following self-duality equation [73, 76]

$$[D_A, D_B] = \pm \frac{1}{2} \varepsilon_{AB}{}^{CD} [D_C, D_D]. \tag{A.30}$$

[23]Since we imposed the vanishing of (anti-)self-dual spin connections, $\omega_M^{(+)a} = 0$ or $\omega_M^{(-)a} = 0$, a remaining symmetry is $SU(2)_{L,R}$ up to a rigid rotation. Together with the function λ, so totally four free parameters, it is enough to achieve the condition $\mathcal{L}_{E_A} \mathfrak{v} = 0$.

Conversely one can proceed with precisely reverse order to show that the vector fields $\{D_A\}$ satisfying eq. (A.30) describe the self-dual spin connections satisfying eq. (A.23). Note that the vector fields D_A now preserve a new volume form $\mathfrak{v}_4 = \lambda^{-2}\mathfrak{v}_g$ which can be seen as follows

$$0 = \mathcal{L}_{E_A}(\lambda^{-1}\mathfrak{v}_g) = d\big(\iota_{E_A}(\lambda^{-1}\mathfrak{v}_g)\big) = d\big(\iota_{\lambda E_A}(\lambda^{-2}\mathfrak{v}_g)\big) = d\big(\iota_{D_A}\mathfrak{v}_4\big) = \mathcal{L}_{D_A}\mathfrak{v}_4. \tag{A.31}$$

The function λ in terms of $\mathfrak{v}_4$ is therefore given by

$$\lambda^2 = \mathfrak{v}_4(D_1, D_2, D_3, D_4) \tag{A.32}$$

and the metric is determined by eq. (A.3) as

$$ds^2 = \lambda^2 \delta_{AB} D^A \otimes D^B = \lambda^2 \delta_{AB} D^A_M D^B_N \, dX^M \otimes dX^N \tag{A.33}$$

where $E^A = \lambda D^A$.

In summary eqs.(A.22), (A.23) and (A.30) are equivalent each other (up to a gauge choice) and equally describe self-dual Einstein gravity.

Now eq. (A.30) clearly exposes to us that the self-dual Einstein gravity looks very much like the self-duality equation in gauge theory. Indeed one can easily see from eq. (3.26) that the self-dual Einstein gravity in the form of eq. (A.30) appears as the leading order of the self-dual NC gauge fields described by

$$\widehat{F}_{AB} = \pm\frac{1}{2}\varepsilon_{AB}{}^{CD}\widehat{F}_{CD}. \tag{A.34}$$

This completes the proof of the equivalence between self-dual NC electromagnetism on $\mathbf{R}^4_{NC}$ or $\mathbf{R}^2_C \times \mathbf{R}^2_{NC}$ and self-dual Einstein gravity.

B Einstein equations from gauge fields

In this section we will generalize the equivalence between the emergent gravity and the Einstein gravity to arbitrary NC gauge fields. We show that the dynamics of NC U(1) gauge fields at a commutative limit can be understood as the Einstein gravity described by eq. (3.38) where the energy momentum tensor is given by usual Maxwell fields and by an unusual "Liouville" field related to the conformal factor (or the size of spacetime) given by eq. (3.48). In the end, we will find some remarkable physics regarding to a novel structure of spacetime.

In a non-coordinate (anholonomic) basis $\{E_A\}$ satisfying the commutation relation (A.11), the spin connections $\omega_A{}^B{}_C$ are defined by

$$\nabla_A E_C = \omega_A{}^B{}_C E_B \tag{B.1}$$

where $\nabla_A \equiv \nabla_{E_A}$ is the covariant derivative in the direction of a vector field E_A. Acting on the dual basis $\{E^A\}$, they are given by

$$\nabla_A E^B = -\omega_A{}^B{}_C E^C. \tag{B.2}$$

Since we will impose the torsion free condition, i.e.,

$$T(A,B) = \nabla_{[A}E_{B]} - [E_A, E_B] = 0, \tag{B.3}$$

the spin connections are related to the structure functions

$$f_{ABC} = -\omega_{ACB} + \omega_{BCA}. \tag{B.4}$$

The Riemann curvature tensors in the basis $\{E_A\}$ are defined by

$$R(A,B) = [\nabla_A, \nabla_B] - \nabla_{[A,B]} \tag{B.5}$$

or in component form

$$\begin{aligned} R_{AB}{}^C{}_D &= \langle E^C, R(E_A, E_B)E_D \rangle \\ &= E_A\omega_B{}^C{}_D - E_B\omega_A{}^C{}_D + \omega_A{}^C{}_E\omega_B{}^E{}_D - \omega_B{}^C{}_E\omega_A{}^E{}_D + f_{AB}{}^E\omega_E{}^C{}_D. \end{aligned} \tag{B.6}$$

Imposing the condition that the metric (3.41) is covariantly constant, i.e.,

$$\nabla_C\Big(\eta_{AB}E^A \otimes E^B\Big) = 0, \tag{B.7}$$

or, equivalently,

$$\omega_{CAB} = -\omega_{CBA}, \tag{B.8}$$

the spin connections ω_{CAB} then have the same number of components as f_{ABC}. Thus eq. (B.4) has a unique solution and it is precisely given by eq. (A.9). In coordinate (holonomic) bases $\{\partial_M, dX^M\}$, the curvature tensors (B.6) also coincide with eq. (A.8). The definition (B.5) together with the metricity condition (B.8) immediately leads to the following symmetry property

$$R_{ABCD} = -R_{ABDC} = -R_{BACD}. \tag{B.9}$$

As we remarked in section 3.2, we want to represent the Riemann curvature tensors in eq. (B.6) in terms of the gauge theory basis D_A in order to use the equations of motion (3.51) and the Bianchi identity (3.52). Using the relation (3.43), the spin connections in eq. (A.9) are given by

$$\lambda\omega_{ABC} = \frac{1}{2}(\mathfrak{f}_{ABC} - \mathfrak{f}_{BCA} + \mathfrak{f}_{CAB}) - D_B \log \lambda\eta_{CA} + D_C \log \lambda\eta_{AB}. \tag{B.10}$$

It is then straightforward to calculate each term in eq. (B.6). We list the results:

$$
\begin{aligned}
E_A\omega_{BCD} &= -\frac{1}{2\lambda^2}D_A\log\lambda(\mathfrak{f}_{BCD}-\mathfrak{f}_{CDB}+\mathfrak{f}_{DBC}) \\
&\quad+\frac{1}{\lambda^2}\eta_{BD}D_A\log\lambda D_C\log\lambda-\frac{1}{\lambda^2}\eta_{BC}D_A\log\lambda D_D\log\lambda \\
&\quad+\frac{1}{2\lambda^2}D_A(\mathfrak{f}_{BCD}-\mathfrak{f}_{CDB}+\mathfrak{f}_{DBC}) \\
&\quad+\frac{1}{\lambda^2}\Big(\eta_{BC}D_AD_D\log\lambda-\eta_{BD}D_AD_C\log\lambda\Big), \qquad\text{(B.11)}\\
\omega_{ACE}\omega_B{}^E{}_D &= \frac{1}{4\lambda^2}\eta^{EF}(\mathfrak{f}_{ACE}-\mathfrak{f}_{CEA}+\mathfrak{f}_{EAC})(\mathfrak{f}_{BFD}-\mathfrak{f}_{FDB}+\mathfrak{f}_{DBF}) \\
&\quad+\frac{1}{2\lambda^2}\eta^{EF}\Big(\eta_{AC}(\mathfrak{f}_{BED}-\mathfrak{f}_{EDB}+\mathfrak{f}_{DBE})-\eta_{BD}(\mathfrak{f}_{ACE}-\mathfrak{f}_{CEA}+\mathfrak{f}_{EAC})\Big)D_F\log\lambda \\
&\quad+\frac{1}{2\lambda^2}\Big((\mathfrak{f}_{ACB}-\mathfrak{f}_{CBA}+\mathfrak{f}_{BAC})D_D\log\lambda-(\mathfrak{f}_{BAD}-\mathfrak{f}_{ADB}+\mathfrak{f}_{DBA})D_C\log\lambda\Big) \\
&\quad+\frac{1}{\lambda^2}\Big(\eta_{BD}D_A\log\lambda D_C\log\lambda-\eta_{AB}D_C\log\lambda D_D\log\lambda+\eta_{AC}D_B\log\lambda D_D\log\lambda\Big) \\
&\quad-\frac{1}{\lambda^2}\eta_{AC}\eta_{BD}\eta^{EF}D_E\log\lambda D_F\log\lambda, \qquad\text{(B.12)}\\
f_{AB}{}^E\omega_{ECD} &= \frac{1}{2\lambda^2}\mathfrak{f}_{AB}{}^E(\mathfrak{f}_{ECD}-\mathfrak{f}_{CDE}+\mathfrak{f}_{DEC}) \\
&\quad+\frac{1}{\lambda^2}(\mathfrak{f}_{ABC}D_D\log\lambda-\mathfrak{f}_{ABD}D_C\log\lambda) \\
&\quad+\frac{1}{2\lambda^2}\Big((\mathfrak{f}_{BCD}-\mathfrak{f}_{CDB}+\mathfrak{f}_{DBC})D_A\log\lambda-(\mathfrak{f}_{ACD}-\mathfrak{f}_{CDA}+\mathfrak{f}_{DAC})D_B\log\lambda\Big) \\
&\quad+\frac{1}{\lambda^2}\Big(\eta_{BC}D_A\log\lambda D_D\log\lambda-\eta_{BD}D_A\log\lambda D_C\log\lambda\Big) \\
&\quad+\frac{1}{\lambda^2}\Big(\eta_{AD}D_B\log\lambda D_C\log\lambda-\eta_{AC}D_B\log\lambda D_D\log\lambda\Big). \qquad\text{(B.13)}
\end{aligned}
$$

Substituting these expressions into eq. (B.6), the curvature tensors are given by

$$
\begin{aligned}
R_{ABCD} = \frac{1}{\lambda^2}\Bigg[\Bigg\{&\frac{1}{2}D_A(\mathfrak{f}_{BCD}-\mathfrak{f}_{CDB}+\mathfrak{f}_{DBC}) \qquad\text{(B.14)}\\
&+\eta_{BC}D_AD_D\log\lambda-\eta_{BD}D_AD_C\log\lambda \\
&+\frac{1}{4}\eta^{EF}(\mathfrak{f}_{ACE}-\mathfrak{f}_{CEA}+\mathfrak{f}_{EAC})(\mathfrak{f}_{BFD}-\mathfrak{f}_{FDB}+\mathfrak{f}_{DBF}) \\
&+\frac{1}{2}\eta^{EF}\Big(\eta_{AC}(\mathfrak{f}_{BED}-\mathfrak{f}_{EDB}+\mathfrak{f}_{DBE})-\eta_{BD}(\mathfrak{f}_{ACE}-\mathfrak{f}_{CEA}+\mathfrak{f}_{EAC})\Big)D_F\log\lambda \\
&+\frac{1}{2}\Big((\mathfrak{f}_{ACB}-\mathfrak{f}_{CBA}+\mathfrak{f}_{BAC})D_D\log\lambda-(\mathfrak{f}_{BAD}-\mathfrak{f}_{ADB}+\mathfrak{f}_{DBA})D_C\log\lambda\Big) \\
&+\eta_{BD}D_A\log\lambda D_C\log\lambda+\eta_{AC}D_B\log\lambda D_D\log\lambda \\
&-\eta_{AC}\eta_{BD}\eta^{EF}D_E\log\lambda D_F\log\lambda\Bigg\}-\{A\leftrightarrow B\}\Bigg] \\
&+\frac{1}{\lambda^2}\Big[\frac{1}{2}\mathfrak{f}_{AB}{}^E(\mathfrak{f}_{ECD}-\mathfrak{f}_{CDE}+\mathfrak{f}_{DEC})+(\mathfrak{f}_{ABC}D_D\log\lambda-\mathfrak{f}_{ABD}D_C\log\lambda)\Big].
\end{aligned}
$$

Using eq. (B.14), the Ricci tensors $R_{AC} \equiv \eta^{BD} R_{ABCD}$ and the Ricci scalar $R \equiv \eta^{AC} R_{AC}$ are accordingly determined as

$$\begin{aligned}
R_{AC} = \frac{1}{\lambda^2}\Big[& -\frac{1}{2}(D-4)(D_A D_C + D_C D_A)\log\lambda - \eta_{AC}\eta^{BD} D_B D_D \log\lambda \\
& +(D-2) D_A \log\lambda D_C \log\lambda - (D-4)\eta_{AC}\eta^{BD} D_B \log\lambda D_D \log\lambda \\
& +\frac{1}{2}(D-4)\eta^{BD}\big(\mathfrak{f}_{ABC} - \mathfrak{f}_{BCA}\big) D_D \log\lambda \\
& -\frac{1}{2}\eta^{BD} D_B(\mathfrak{f}_{ACD} - \mathfrak{f}_{CDA} + \mathfrak{f}_{DAC}) \\
& +\frac{1}{4}\eta^{BD}\eta^{EF}\mathfrak{f}_{BEC}\mathfrak{f}_{DFA} + \frac{1}{2}\eta^{BD}\mathfrak{f}_{AB}{}^{E}(\mathfrak{f}_{ECD} - \mathfrak{f}_{CDE})\Big],
\end{aligned} \tag{B.15}$$

$$\begin{aligned}
R = \frac{1}{\lambda^2}\Big[& -2(D-3)\eta^{AC} D_A D_C \log\lambda - (D-2)(D-5)\eta^{AC} D_A \log\lambda D_C \log\lambda \\
& +\frac{1}{4}\eta^{AC}\eta^{BD}\mathfrak{f}_{AB}{}^{E}(2\mathfrak{f}_{ECD} - \mathfrak{f}_{CDE})\Big],
\end{aligned} \tag{B.16}$$

where we have used the relation (3.46) and

$$\begin{aligned}
& \frac{1}{4}\eta^{BD}\eta^{EF}(\mathfrak{f}_{BCE} - \mathfrak{f}_{CEB} + \mathfrak{f}_{EBC})(\mathfrak{f}_{AFD} - \mathfrak{f}_{FDA} + \mathfrak{f}_{DAF}) \\
& \qquad = \frac{1}{2}\eta^{BD}\mathfrak{f}_{AB}{}^{E}\mathfrak{f}_{DEC} - \frac{1}{4}\eta^{BD}\eta^{EF}\mathfrak{f}_{BEC}\mathfrak{f}_{DFA}.
\end{aligned} \tag{B.17}$$

Up to now we have not used eqs.(3.51) and (3.52). We have simply calculated curvature tensors for an arbitrary metric (3.41). Now we will impose on the curvature tensors the equations of motion eq. (3.51) and the Bianchi identity (3.52). First note the following identity

$$\begin{aligned}
& R(E_A, E_B)E_C + R(E_B, E_C)E_A + R(E_C, E_A)E_B \\
& \qquad = [E_A, [E_B, E_C]] + [E_B, [E_C, E_A]] + [E_C, [E_A, E_B]]
\end{aligned} \tag{B.18}$$

which can be derived using the condition (B.3). The Jacobi identity then implies $R_{[ABC]D} = 0$. Since $D_A = \lambda E_A$, we have the relation $[D_{[A}, [D_B, D_{C]}]] = \lambda^3 [E_{[A}, [E_B, E_{C]}]]$ where all the terms containing the derivations of λ cancel each other. Thus the first Bianchi identity $R_{[ABC]D} = 0$ follows from the Jacobi identity $[D_{[A}, [D_B, D_{C]}]] = 0$. Then eq. (3.52) confirms that the guess (3.37) is pleasingly true, i.e.,

$$\widehat{D}_{[A}\widehat{F}_{BC]} = 0 \quad \Longleftrightarrow \quad R_{[ABC]D} = 0. \tag{B.19}$$

One can also directly check eq. (B.19) using the expression (B.14):

$$R_{[ABC]D} = \frac{1}{\lambda^2}\Big(D_{[A}\mathfrak{f}_{BC]D} - \mathfrak{f}_{[BC}{}^{E}\mathfrak{f}_{A]ED}\Big) = 0. \tag{B.20}$$

Let us summarize the algebraic symmetry of curvature tensors determined by the properties about the torsion and the tangent-space group:

$$R_{ABCD} = -R_{ABDC} = -R_{BACD}, \tag{B.21}$$

$$R_{[ABC]D} = 0, \tag{B.22}$$

$$R_{ABCD} = R_{CDAB} \tag{B.23}$$

where the last symmetry can be derived by using the others. Therefore it is obvious that the vector fields $D_A \in TM$ satisfying eq. (3.52) describe a usual (pseudo-)Riemannian manifold.

Some useful properties can be further deduced. Contracting the indices C and D in eq. (3.52) leads to

$$D_A\rho_B - D_B\rho_A + \mathfrak{f}_{AB}{}^C\rho_C = D_C\mathfrak{f}_{AB}{}^C \tag{B.24}$$

and the left-hand side identically vanishes due to eq. (3.40) with eq. (3.46). Thus we get

$$D_C\mathfrak{f}_{AB}{}^C = 0. \tag{B.25}$$

Similarly, from eq. (3.51), we get

$$\eta^{AB}D_AD_B\log\lambda = \frac{1}{2}D_A\rho^A = -\frac{1}{2}\eta^{AB}\mathfrak{f}_{AC}{}^D\mathfrak{f}_{BD}{}^C. \tag{B.26}$$

eq. (B.25) now guarantees that the Ricci tensor (B.15) is symmetric, i.e., $R_{AC} = R_{CA}$. (It should be the case since the symmetry property (B.23) shows that $R_{AC} = \eta^{BD}R_{ABCD} = \eta^{DB}R_{CDAB} = R_{CA}$. Recall that the property (B.23) results from the Bianchi identity (B.20).)

In order to check the conjecture (3.38), we first consider the Euclidean $D = 4$ case since we already know the answer for the self-dual case. For the Euclidean space we will not care about raising and lowering indices. Using eqs.(3.46), (3.51) and (B.26), the Ricci tensor (B.15) can be rewritten as follows

$$\begin{aligned} R_{AC} = \frac{1}{2\lambda^2}\Big[&\delta_{AC}\mathfrak{f}_{BDE}\mathfrak{f}_{BED} + \mathfrak{f}_{BAB}\mathfrak{f}_{DCD} - \mathfrak{f}_{BDA}\mathfrak{f}_{BCD} - \mathfrak{f}_{BDC}\mathfrak{f}_{BAD} \\ &+\frac{1}{2}\mathfrak{f}_{BDA}\mathfrak{f}_{BDC} + \mathfrak{f}_{ABD}\mathfrak{f}_{DCB} - \mathfrak{f}_{ABD}\mathfrak{f}_{CBD}\Big]. \end{aligned} \tag{B.27}$$

Now we decompose $\mathfrak{f}_{ABC}$ into self-dual and anti-self-dual parts as in eq. (A.12)

$$\mathfrak{f}_{ABC} = \mathfrak{f}_C^{(+)a}\eta^a_{AB} + \mathfrak{f}_C^{(-)a}\bar{\eta}^a_{AB} \tag{B.28}$$

where

$$\mathfrak{f}_C^{(\pm)a}\eta^{(\pm)a}_{AB} = \frac{1}{2}\Big(\mathfrak{f}_{ABC} \pm \frac{1}{2}\varepsilon_{AB}{}^{DE}\mathfrak{f}_{DEC}\Big) \tag{B.29}$$

and introduce a completely antisymmetric tensor defined by

$$\Psi_{ABC} = \mathfrak{f}_{ABC} + \mathfrak{f}_{BCA} + \mathfrak{f}_{CAB} \equiv \varepsilon_{ABCD}\Psi_D. \tag{B.30}$$

Using the decomposition (B.28) and eq. (A.15) one can easily see that

$$\Psi_A = -\frac{1}{3!}\varepsilon_{ABCD}\Psi_{BCD} = -(\mathfrak{f}_B^{(+)a}\eta^a_{AB} - \mathfrak{f}_B^{(-)a}\bar{\eta}^a_{AB}), \tag{B.31}$$

while eq. (3.46) leads to

$$\rho_A = \mathfrak{f}_{BAB} = \mathfrak{f}_B^{(+)a}\eta^a_{AB} + \mathfrak{f}_B^{(-)a}\bar{\eta}^a_{AB}. \tag{B.32}$$

JHEP05(2009)012

The calculation of the Ricci tensor (B.27) can straightforwardly be done using the decomposition (B.28) and the identities (A.19) and (A.20) after rewriting the following term

$$\begin{aligned}
\mathfrak{f}_{ABD}\mathfrak{f}_{DCB} &= \mathfrak{f}_{ABD}(\Psi_{DCB} - \mathfrak{f}_{CBD} - \mathfrak{f}_{BDC}) \\
&= \varepsilon_{DCBE}(\mathfrak{f}_D^{(+)a}\eta^a_{AB} + \mathfrak{f}_D^{(-)a}\bar{\eta}^a_{AB})\Psi_E - \mathfrak{f}_{ABD}\mathfrak{f}_{CBD} - \mathfrak{f}_{ABD}\mathfrak{f}_{BDC} \\
&= -\Psi_A\Psi_C - (\mathfrak{f}_A^{(+)a}\eta^a_{CD} - \mathfrak{f}_A^{(-)a}\bar{\eta}^a_{CD})\Psi_D + \delta_{AC}\Psi_D\Psi_D \\
&\quad -\mathfrak{f}_{ABD}\mathfrak{f}_{CBD} - \mathfrak{f}_{ABD}\mathfrak{f}_{BDC}
\end{aligned} \tag{B.33}$$

where eq. (A.17) was used at the last step. An interesting thing is that eq. (B.33) cancels most terms in eq. (B.27) leaving a remarkably simple form

$$\begin{aligned}
R_{AC} = -\frac{1}{\lambda^2}\Big[&\mathfrak{f}_D^{(+)a}\eta^a_{AB}\mathfrak{f}_D^{(-)b}\bar{\eta}^b_{CB} + \mathfrak{f}_D^{(+)a}\eta^a_{CB}\mathfrak{f}_D^{(-)b}\bar{\eta}^b_{AB} \\
&-\big(\mathfrak{f}_B^{(+)a}\eta^a_{AB}\mathfrak{f}_D^{(-)b}\bar{\eta}^b_{CD} + \mathfrak{f}_B^{(+)a}\eta^a_{CB}\mathfrak{f}_D^{(-)b}\bar{\eta}^b_{AD}\big)\Big].
\end{aligned} \tag{B.34}$$

Note that the right-hand side of eq. (B.34) is purely interaction terms between the self-dual and anti-self-dual parts in eq. (B.28). (The same result was also obtained in [75].) Therefore, if NC gauge fields satisfy the self-duality equation (3.44), they describe a Ricci-flat manifold, i.e., $R_{AC} = 0$. Of course, this result is completely consistent with that in appendix A. Moreover we see the reason why self-dual NC gauge fields satisfy the Einstein equation (3.38) with vanishing energy-momentum tensor.

Finally we can calculate the Einstein tensor to find the form of the energy-momentum tensor defined by eq. (3.38):

$$\begin{aligned}
E_{AB} &= R_{AB} - \frac{1}{2}\delta_{AB}R \\
&= -\frac{1}{\lambda^2}\Big(\mathfrak{f}_D^{(+)a}\eta^a_{AC}\mathfrak{f}_D^{(-)b}\bar{\eta}^b_{BC} + \mathfrak{f}_D^{(+)a}\eta^a_{BC}\mathfrak{f}_D^{(-)b}\bar{\eta}^b_{AC}\Big) \\
&\quad +\frac{1}{\lambda^2}\Big(\mathfrak{f}_C^{(+)a}\eta^a_{AC}\mathfrak{f}_D^{(-)b}\bar{\eta}^b_{BD} + \mathfrak{f}_C^{(+)a}\eta^a_{BC}\mathfrak{f}_D^{(-)b}\bar{\eta}^b_{AD} - \delta_{AB}\mathfrak{f}_D^{(+)a}\eta^a_{CD}\mathfrak{f}_E^{(-)b}\bar{\eta}^b_{CE}\Big)
\end{aligned} \tag{B.35}$$

where the Ricci scalar R is given by

$$R = \frac{2}{\lambda^2}\mathfrak{f}_B^{(+)a}\eta^a_{AB}\mathfrak{f}_C^{(-)b}\bar{\eta}^b_{AC}. \tag{B.36}$$

We have adopted the conventional view that the gravitational field is represented by the spacetime metric itself. The problem then becomes one of finding field equations to relate the metric (3.41) to the energy-momentum distribution. According to our scheme, eq. (B.35) should correspond to such field equations, i.e., the Einstein equations. In other words, if we are clever enough, we should be able to find the NC gauge theory described by eqs.(3.51) and (3.52) starting from the Einstein gravity described by eqs.(B.22) and (B.35) by properly reversing our above derivation as we have explicitly demonstrated it for the self-dual case in appendix A.

As we explained in section 3.2, we want to identify eq. (B.35) with an energy-momentum tensor. First note that the Ricci scalar R, (B.36), is nonvanishing for a generic

JHEP05(2009)012

case. This means that there is an extra field contribution to the energy-momentum tensor in addition to Maxwell fields whose energy-momentum tensor is traceless. Since the extra field energy-momentum tensor turns out to be basically a gradient volume energy (see the latter part of section 3.2), we call it the "Liouville" energy-momentum tensor. A similar result was also obtained in [17] where it was dubbed as the 'Poisson' energy. Since the first term in eq. (B.35) is traceless due to eq. (A.18), it would be a candidate of the Maxwell energy-momentum tensor while the second term would be the Liouville energy-momentum tensor. So we tentatively make the following identification for the Maxwell energy-momentum tensor $T^{(M)}_{AB}$ and the Liouville energy-momentum tensor $T^{(L)}_{AB}$

$$\begin{aligned}
\frac{8\pi G_4}{c^4}T^{(M)}_{AB} &= -\frac{1}{\lambda^2}\Big(\mathfrak{f}^{(+)a}_D\eta^a_{AC}\mathfrak{f}^{(-)b}_D\bar{\eta}^b_{BC}+\mathfrak{f}^{(+)a}_D\eta^a_{BC}\mathfrak{f}^{(-)b}_D\bar{\eta}^b_{AC}\Big),\\
&= -\frac{1}{\lambda^2}\Big(\mathfrak{f}_{ACD}\mathfrak{f}_{BCD}-\frac{1}{4}\delta_{AB}\mathfrak{f}_{CDE}\mathfrak{f}_{CDE}\Big),
\end{aligned} \tag{B.37}$$

$$\begin{aligned}
\frac{8\pi G_4}{c^4}T^{(L)}_{AB} &= \frac{1}{\lambda^2}\Big(\mathfrak{f}^{(+)a}_C\eta^a_{AC}\mathfrak{f}^{(-)b}_D\bar{\eta}^b_{BD}+\mathfrak{f}^{(+)a}_C\eta^a_{BC}\mathfrak{f}^{(-)b}_D\bar{\eta}^b_{AD}-\delta_{AB}\mathfrak{f}^{(+)a}_D\eta^a_{CD}\mathfrak{f}^{(-)b}_E\bar{\eta}^b_{CE}\Big),\\
&= \frac{1}{2\lambda^2}\Big(\rho_A\rho_B-\Psi_A\Psi_B-\frac{1}{2}\delta_{AB}(\rho_C^2-\Psi_C^2)\Big)
\end{aligned} \tag{B.38}$$

where we have used the decomposition (B.29) and the relation

$$\mathfrak{f}^{(+)a}_B\eta^a_{AB}=\frac{1}{2}(\rho_A-\Psi_A), \qquad \mathfrak{f}^{(-)a}_B\bar{\eta}^a_{AB}=\frac{1}{2}(\rho_A+\Psi_A).$$

We have anticipated that the energy-momentum tensor (B.37) will be related to that of Maxwell fields since both are definitely traceless in four dimensions. So our problem is how to rewrite the energy-momentum tensor in terms of NC fields in $\star$-algebra $\mathcal{A}_\theta$, denoted as $\widehat{T}_{AB}(\mathcal{A}_\theta)$, using the expression (B.37) defined in TM, denoted as $T_{AB}(TM)$. In other words, we want to translate $T_{AB}(TM)$ into an $\mathcal{A}_\theta$-valued energy momentum tensor. This problem is quite subtle.

Recall that NC fields are identified with vector fields in TM through the map (3.23) at the leading order. For example, we get the following identification from eq. (3.26)

$$\begin{aligned}
-i[\widehat{F}_{AB},\widehat{f}]_\star &= \{F_{AB},f\}_\theta+\cdots=[D_A,D_B][f]+\cdots\\
&= -\mathfrak{f}_{AB}{}^C D_C[f]+\cdots.
\end{aligned} \tag{B.39}$$

Note that eq. (B.39) is nothing but the Lie algebra homomorphism (3.29) for the Poisson algebra. But a NC field regarded as an element of NC $\star$-algebra $\mathcal{A}_\theta$ in general lives in a Hilbert space $\mathcal{H}$, e.g., the Fock space (3.2) while the vector fields D_A in eq. (3.23) are defined in the real vector space TM. Furthermore we see from eqs.(3.23) and (B.39) that "anti-Hermitian" operators in NC algebra $\mathcal{A}_\theta$ such as the NC fields $\widehat{D}_A$ and $-i\widehat{F}_{AB}$ are mapped to real vector fields in TM. Thus we have the bizarre correspondence between geometry defined in TM and NC algebra $\mathcal{A}_\theta$[24]

$$\text{Anti}-\text{Hermitian operators on }\mathcal{H} \quad\Leftrightarrow\quad \text{Real vector fields on }TM. \tag{B.40}$$

[24]It might be remarked that the transition from TM to $\mathcal{A}_\theta$ is analogous to that from classical mechanics (an **R**-world) to quantum mechanics (a **C**-world). See section 5.4 in [32] for the exposition of the similar problem in the context of quantum mechanics.

JHEP05(2009)012

In order to identify $\mathcal{A}_\theta$-valued quantities from TM-valued ones, it is first necessary to analytically continue the real vector space TM to a complex vector space $TM_{\mathbf{C}}$. At the same time, the real vector field D_A is replaced by a self-adjoint operator $\mathcal{D}_A$ in $TM_{\mathbf{C}}$ and the structure equation (3.40) instead has the form

$$[\mathcal{D}_A, \mathcal{D}_B] = i\mathfrak{f}_{AB}{}^C \mathcal{D}_C. \tag{B.41}$$

Now we want to translate a quantity defined on $TM_{\mathbf{C}}$ such as eq. (B.41) into a NC field defined on $\mathcal{H}$ as the Weyl-Wigner correspondence [1]. Since we have the identification (B.39), we need to relate the inner product on the operator algebra $\mathcal{A}_\theta$, denoted as $\langle \widehat{V}, \widehat{W}\rangle_{\mathcal{A}_\theta}$ for $\widehat{V}, \widehat{W} \in \mathcal{A}_\theta$ to the inner product $\langle V, W\rangle_{TM_{\mathbf{C}}} \equiv \overline{V} \cdot W$ on $TM_{\mathbf{C}}$ for $V, W \in TM_{\mathbf{C}}$, both of which are defined to be positive definite. To do this, we will take the natural prescription according to the correspondence (B.39)

$$\langle \widehat{F}_{AB}, \widehat{F}_{CD}\rangle_{\mathcal{A}_\theta} \;\Leftrightarrow\; \mathfrak{f}_{AB}{}^E \mathfrak{f}_{CD}{}^F (\mathcal{D}_E \cdot \mathcal{D}_F) + \cdots \tag{B.42}$$

where the ellipsis means that we need a general inner product for multi-indexed vector fields, e.g., polyvector fields though the leading term is enough for our purpose. Note that $\mathcal{D}_A = \lambda \mathcal{E}_A$ carry the mass dimension, i.e., $[\mathcal{D}_A] = [\mathcal{E}_A] = L^{-1}$ where λ is chosen to be real such that both $\mathcal{D}_A$ and $\mathcal{E}_A$ are self-adjoint operators in $TM_{\mathbf{C}}$. Hence we will take into account the physical dimension of the vector fields $\mathcal{D}_A$ in the definition of the inner product (B.42)

$$\mathcal{D}_A \cdot \mathcal{D}_B = \lambda^2 (\mathcal{E}_A \cdot \mathcal{E}_B) = \frac{\lambda^2}{|\mathrm{Pf}\theta|^{\frac{1}{n}}} \delta_{AB}. \tag{B.43}$$

Here the noncommutativity $|\theta|$ is the most natural dimensionful parameter at our hands that can enter the definition (B.43).

Suppose that the analytic continuation was performed and we adopt the prescription (B.42). Then the analytic continuation from TM to $TM_{\mathbf{C}}$ accompanies the i factor in the structure equation (B.41) which will introduce a sign flip in eq. (B.37).[25] And then $T_{AB}(TM_{\mathbf{C}})$ will be identified using the prescription (B.42) with $\widehat{T}_{AB}(\mathcal{A}_\theta)$. After taking the sign flip into account, one can finally identify $\widehat{T}_{AB}(\mathcal{A}_\theta)$ from the Maxwell energy-momentum tensor (B.37)

$$\frac{8\pi G_4}{c^4} \widehat{T}^{(M)}_{AB}(\mathcal{A}_\theta) = \frac{g_{YM}^2 |\mathrm{Pf}\theta|^{\frac{1}{n}}}{\hbar^2 c^2 \lambda^4} \frac{\hbar^2 c^2}{g_{YM}^2} \left(\widehat{F}_{AC} \widehat{F}_{BC} - \frac{1}{4} \delta_{AB} \widehat{F}_{CD} \widehat{F}_{CD} \right) \tag{B.44}$$

where we simply rewrote the global factor for later use. Recall that we are taking the commutative limit $|\theta| \to 0$ (see the paragraph in eq. (3.39)). Thus one can simply replace the field strengths in eq. (B.44) by commutative ones, i.e., $\widehat{F}_{AC} \approx F_{AC} + \mathcal{O}(\theta)$, since the

[25] To avoid any confusion, we point out that it never means changing the sign of eq. (B.37) because eq. (B.37) is obviously defined on TM. It simply prescribes the analytic continuation to get a correct definition of $\widehat{T}_{AB}(\mathcal{A}_\theta)$. Anyway we think that this perverse sign problem will disappear (at the price of transparent geometrical picture) if we work in the vector space $TM_{\mathbf{C}}$ from the outset using the structure equation (B.41). It will also be useful to clearly understand the structure of Hilbert space defining (quantum) gravity, especially, in the context of emergent gravity. We hope to address this approach in the near future.

JHEP05(2009)012

global factor $|\mathrm{Pf}\theta|^{\frac{1}{n}}$ already contains $\mathcal{O}(\theta)$. Therefore, in the commutative limit, the trace of NC spacetime in eq. (B.44) only remains in the global factor which will be identified with the Newton constant. Thus we get the usual Maxwell energy-momentum tensor at the leading order. It should be pointed out that the energy momentum tensor (B.44) is not quite the same as that derived from the action (3.9) since the background part B_{MN} does not appear in the result. We will see in section 3.4 that this fact bears an important consequence about the cosmological constant and dark energy.

Note that the result (B.44) is independent of spacetime dimensions including the front factor. By comparing the expression (B.44) with eq. (3.38), we get the identification of the Newton "constant"

$$G_D = \frac{c^2 g_{YM}^2 |\mathrm{Pf}\theta|^{\frac{1}{n}}}{8\pi\hbar^2\lambda^4}. \tag{B.45}$$

Thereby we almost confirmed eq. (3.39) obtained by a simple dimensional analysis except the dimensionless factor λ^4. (Of course the dimensional analysis alone cannot fix any dimensionless parameters.) Then eq. (B.45) comes with a surprise. It raises a question whether the Newton "constant" G_D is a constant or not. If it is a constant, then it means that g_{YM} (or even $\hbar$ and c ?) depends on λ such that G_D is a constant. Or if g_{YM}, c and $\hbar$ are really constants, G_D depends on the conformal factor (or the size of spacetime) given by eq. (3.48). We prefer the former interpretation since we know that g_{YM} changes under a renormalization group flow. Furthermore we note that g_{YM}^2 in NC gauge theory depends on an open string metric in B-field background [22] and λ^2 is also related to the metric g_{MN} through the relation (3.48). (In four dimensions $\lambda^2 \sim \sqrt{-g}$.) Nevertheless, we could not find any inconsistency for the latter interpretation either, because it seems to be consistent with current laboratory experiments since $\lambda = 1$ for any flat spacetime.

In the course of our derivation, we have introduced a completely antisymmetric tensor

$$\Psi_{ABC} = \mathfrak{f}_{ABC} + \mathfrak{f}_{BCA} + \mathfrak{f}_{CAB}. \tag{B.46}$$

So one may identify it with a 3-form field

$$H \equiv \frac{1}{3!}\Psi_{ABC}E^A \wedge E^B \wedge E^C = \frac{\lambda}{2} f_{ABC}E^A \wedge E^B \wedge E^C \tag{B.47}$$

where we used eq. (3.43). But H is not a closed 3-form in general. Using the structure equation

$$dE^A = \frac{1}{2} f_{BC}{}^A E^B \wedge E^C \tag{B.48}$$

one can show that instead it satisfies the following relation

$$\begin{aligned} dH &= \frac{\lambda}{2}(E_A f_{BCD} - f_{BC}{}^E f_{AED})E^A \wedge E^B \wedge E^C \wedge E^D \\ &\quad + \left(\frac{1}{4\lambda}\mathfrak{f}_{AD}{}^E \mathfrak{f}_{BCE} + \frac{3\lambda}{2}E_A \log\lambda f_{BCD}\right)E^A \wedge E^B \wedge E^C \wedge E^D \\ &= \frac{|\mathrm{Pf}\theta|^{\frac{1}{n}}}{\lambda^3} F \wedge F + 3d\log\lambda \wedge H \end{aligned} \tag{B.49}$$

where we used the Jacobi identity $[E_{[A},[E_B,E_{C]}]]=0$ to show the vanishing of the first term and the map (B.42) for the second term. From eq. (B.49) we see that $\widetilde{H}\equiv\lambda^{-3}H=\frac{1}{3!}\Psi_{ABC}D^A\wedge D^B\wedge D^C$ is closed, i.e., $d\widetilde{H}=0$, if and only if $F\wedge F=0$. In this case locally $\widetilde{H}=d\widetilde{B}$ by the Poincaré lemma. Indeed the 3-form $\widetilde{H}=d\widetilde{B}$ is quite similar to the Kalb-Ramond field in string theory while the conformal factor λ in eq. (3.46) behaves like a dilaton field in string theory. In its overall picture the emergent gravity is very similar to string theory where a metric g_{MN}, an NS-NS 3-form $H=dB$ and a dilaton Φ describe a gravitational theory in D dimensions.

Now we go to the second energy-momentum tensor (B.38). Note that ρ_A is determined by the volume factor in eq. (3.48) evaluated in the gauge theory basis $\{D_A\}$ while Ψ_A is coming from the 3-form (B.47). eq. (B.38) has an interesting property that they identically vanish for flat spacetime and self-dual gauge fields where $\rho_A=\pm\Psi_A$. This kind of energy has no counterpart in commutative spacetime and would be a unique property appearing only in NC spacetime. This exotic feature might be expected from the beginning because the NC spacetime leads to a perplexing mixing between short (UV) and large (IR) distance scales [55]. To illuminate the property of the energy-momentum tensor (B.38), let us simply assume that its average (in a broad sense) is SO(4) invariant, i.e.,

$$\langle\rho_A\rho_B\rangle=\frac{1}{4}\delta_{AB}\rho_C^2,\quad \langle\Psi_A\Psi_B\rangle=\frac{1}{4}\delta_{AB}\Psi_C^2. \tag{B.50}$$

Then the average of the energy-momentum tensor (B.38) is given by

$$\langle T_{AB}^{(L)}\rangle=-\frac{c^4}{64\pi G_4\lambda^2}\delta_{AB}(\rho_C^2-\Psi_C^2). \tag{B.51}$$

Note that the Ricci scalar (B.36) is purely coming from this source since eq. (B.44) is traceless. For a constant curvature space, e.g., de Sitter or anti-de Sitter space, the Ricci scalar $R=\frac{1}{2\lambda^2}(\rho_A^2-\Psi_A^2)$ will be constant. In this case the energy-momentum tensor (B.51) precisely behaves like a cosmological constant since $T_{AB}^{(L)}=-\frac{c^4}{32\pi G_4}\delta_{AB}R$. Of course this conclusion is meaningful only if eq. (B.35) allows a constant curvature spacetime. But the energy momentum tensor given by eq. (B.51) will behave like a cosmological constant as ever for an almost constant curvature space as shown in eq. (3.96).

Although we have taken the Euclidean signature for convenience, it can be analytically continued to the Lorentzian signature.[26] For example, a crucial step in our approach was the decomposition (B.28). But that decomposition can also be done in the Lorentzian signature by introducing an imaginary self-duality $\eta_{AB}^{(\pm)a}=\pm\frac{i}{2}\varepsilon_{AB}{}^{CD}\eta_{CD}^{(\pm)a}$ where $SU(2)_{L,R}$ is formally extended to $SL(2,\mathbf{C})$. Indeed the proof in appendix A can equally be done using the imaginary self-duality as adopted in [73]. Or equivalently we can use the spinor representation [30] for an arbitrary anti-symmetric rank 2-tensor

$$F_{AB}=F_{ab\dot{a}\dot{b}}=\varepsilon_{\dot{a}\dot{b}}\phi_{ab}+\varepsilon_{ab}\psi_{\dot{a}\dot{b}} \tag{B.52}$$

[26]The Wick rotation will be defined by $x^4=ix^0$. Under this Wick rotation, $\delta_{AB}\to\eta_{AB}=(-+++)$ and $\varepsilon^{1234}=1\to-\varepsilon^{0123}=-1$. Then we get $\Psi_A^{(E)}=i\Psi_A^{(L)}$ according to the definition (B.30).

where $a, \dot{a}, \cdots$ are SL$(2, \mathbf{C})$ spinor indices. For a real 2-form, $\psi = \bar{\phi}$. In this notation, the 2-form dual to F_{AB} is given by

$$
\begin{aligned}
{}^*F_{AB} &= \frac{1}{2}\varepsilon_{AB}{}^{CD}F_{CD} = {}^*F_{ab\dot{a}\dot{b}} && \text{(B.53)}\\
&= -i\varepsilon_{\dot{a}\dot{b}}\phi_{ab} + i\varepsilon_{ab}\psi_{\dot{a}\dot{b}}, && \text{(B.54)}
\end{aligned}
$$

that is,

$$
{}^*F_{ab\dot{a}\dot{b}} = iF_{ab\dot{b}\dot{a}} = -iF_{ba\dot{a}\dot{b}}. \tag{B.55}
$$

For the sake of completeness we will also consider $D = 2$ and $D = 3$ cases. For convenience we consider the Euclidean signature again for both cases. (The $D = 2$ case should be Euclidean in our context since we don't want to consider time-space noncommutativity.) From now on we set $\hbar = c = 1$.

In two dimensions, the analysis is simple. So we immediately list the formulas:

$$
\begin{aligned}
\mathfrak{f}_{ABC} &\equiv \varepsilon_{AB}\Psi_C, && \text{(B.56)}\\
\rho_A &= \mathfrak{f}_{BAB} = 2D_A \log\lambda, && \text{(B.57)}\\
\Psi_A &= \varepsilon_{AB}\rho_B = 2\varepsilon_{AB}D_B \log\lambda, && \text{(B.58)}\\
D_A\rho_A &= -\rho_A\rho_A = -\Psi_A\Psi_A, && \text{(B.59)}\\
D_A\Psi_A &= 0, && \text{(B.60)}\\
R_{ABCD} &= \frac{1}{2}\varepsilon_{AB}\varepsilon_{CD}R = \frac{1}{2}(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC})R, && \text{(B.61)}\\
R &= \frac{2}{\lambda^2}(D_A D_A \log\lambda - 2D_A \log\lambda D_A \log\lambda). && \text{(B.62)}
\end{aligned}
$$

Of course it is a bit lengthy to directly check eq. (B.61) from eq. (B.14).

Using the equation of motion (B.59), the Ricci scalar (B.62) can be rewritten as

$$
R = -\frac{2}{\lambda^2}\rho_A\rho_A = -\frac{2}{\lambda^2}\Psi_A\Psi_A = -\frac{8}{\lambda^2}D_A \log\lambda D_A \log\lambda. \tag{B.63}
$$

The Einstein equation in two dimensions can be written as

$$
R_{AB} = \frac{1}{2}\delta_{AB}R = -\frac{1}{2\lambda^2}\delta_{AB}\mathfrak{f}_{CDE}\mathfrak{f}_{CDE}. \tag{B.64}
$$

An interesting thing in eq. (B.63) is that the Ricci scalar is always negative unlike as the 4-dimensional case where $R = \frac{1}{2\lambda^2}(\rho_A^2 - \Psi_A^2)$. Hence eq. (B.64) describes only hyperbolic (negative curvature) Riemann surfaces but most Riemann surfaces belong to this class.

From eq. (B.64) one can see that the case with $\widehat{F}_{AB} = 0$ corresponds to parabolic (curvature 0) Riemann surfaces which include a plane $\mathbf{R}^2$ and a torus $\mathbf{T}^2$. Then a natural question is where the different topology for $\mathbf{R}^2$ and $\mathbf{T}^2$ comes from. Note that there are still background gauge fields given by eq. (3.1) although the *fluctuations* are vanishing. (Two-dimensional gauge fields do not have any physical degrees of freedom but encode only a topological information. So the *fluctuations* here mean the variation of a topological shape.) We observe that, though $B \in H^2(M)$ in eq. (3.1) is constant, it reveals its topology through the first cohomology group $H^1(M)$ which measures the obstruction for symplectic vector fields to be globally Hamiltonian (see the footnote 3 in [3]). That is the only source we can imagine for the origin of the topology of Riemann surfaces. We believe that the topology of the fluctuation $\widehat{F}_{AB}$ in eq. (B.64) similarly appears in hyperbolic Riemann surfaces with a higher genus. Then a natural question is about a rational (positive curvature) Riemann surface, i.e., $\mathbf{S}^2$. It may be necessary to introduce a mass term as a potential term. We leave it for a future work.[27]

Now we go over to $D = 3$ case. In three dimensions $\mathfrak{f}_{ABC}$ have totally 9 components. We will decompose them into $9 = 1 + 3 + 5$ as follows

$$\mathfrak{f}_{ABC} = \varepsilon_{ABC}\Psi + \varepsilon_{ABD}(\rho_{DC} + \varphi_{DC}) \tag{B.65}$$

where the first term is totally anti-symmetric part like eq. (B.46) and the second term is anti-symmetric, $\rho_{DC} = -\rho_{CD}$, and the third term is symmetric, $\varphi_{DC} = \varphi_{CD}$, and traceless, $\varphi_{CC} = 0$. eq. (3.46) then leads to the relation $\rho_{AB} = \frac{1}{2}\varepsilon_{ABC}\rho_C$. Therefore we get the following decomposition

$$\mathfrak{f}_{ABC} = \varepsilon_{ABC}\Psi + \frac{1}{2}(\delta_{AC}\rho_B - \delta_{BC}\rho_A) + \varepsilon_{ABD}\varphi_{DC}. \tag{B.66}$$

In other words, the symmetric part can be deduced from eq. (B.66) as follows

$$\varphi_{AB} = \frac{1}{2}\varepsilon_{ACD}\mathfrak{f}_{CDB} - \frac{1}{2}\varepsilon_{ABC}\rho_C - \delta_{AB}\Psi. \tag{B.67}$$

Using the variables in eq. (B.66), the equations of motion (3.51) can be written as

$$D_B\mathfrak{f}_{BCA} = -2\delta_{AC}\Psi^2 - \Psi\varphi_{AC} + \frac{1}{4}(\delta_{AC}\rho_B\rho_B - \rho_A\rho_C) \tag{B.68}$$

$$+\frac{3}{2}\varepsilon_{ACB}\Psi\rho_B + \varepsilon_{CBD}\rho_B\varphi_{DA} + \frac{1}{2}\varepsilon_{ACB}\rho_D\varphi_{BD} + \varphi_{AB}\varphi_{CB}. \tag{B.69}$$

Contracting the indices A and C in the above equation leads to the relation

$$D_A\rho_A = 6\Psi^2 - \frac{1}{2}\rho_A\rho_A - \varphi_{AB}\varphi_{AB}. \tag{B.70}$$

Using the above results, it is straightforward though a bit lengthy to calculate the Ricci tensor (B.15)

$$\begin{aligned} R_{AC} = &-\frac{1}{\lambda^2}\Big(\mathfrak{f}_{ABD}\mathfrak{f}_{CBD} - \frac{1}{4}\delta_{AC}\mathfrak{f}_{BED}\mathfrak{f}_{BED}\Big) \\ &+\frac{1}{4\lambda}(\nabla_A\rho_C + \nabla_C\rho_A) + \frac{1}{2\lambda^2}\rho_A\rho_C \end{aligned} \tag{B.71}$$

and the Ricci scalar (B.16)

$$R = \frac{1}{\lambda}\nabla_A\rho_A + \frac{1}{2\lambda^2}\Big(\rho_A\rho_A - 9\Psi^2\Big). \tag{B.72}$$

[27] In this respect, the work [77] by H. Shimada should be interesting. He showed that the topology of a membrane in matrix theory can be captured by a Hamiltonian function defined on a Riemann surface. The Hamiltonian function for a nontrivial Riemann surface is in general given by a Morse function containing several nondegenerate critical points, e.g., a height function, where the topology of a membrane is realized as the Morse topology.

Since the first term in eq. (B.15) is nonvanishing while it was absent in four dimensions, we introduced the covariant derivative of the "Liouville" field ρ_A defined by

$$\nabla_A \rho_C = E_A \rho_C - \omega_A{}^B{}_C \rho_B \tag{B.73}$$

and then we used the following relation derived from eq. (B.10)

$$\nabla_A \rho_C + \nabla_C \rho_A = \frac{1}{\lambda}\Big(D_A \rho_C + D_C \rho_A - (\mathfrak{f}_{ABC} + \mathfrak{f}_{CBA})\rho_B + \delta_{AC}\rho_B\rho_B - \rho_A\rho_C\Big). \tag{B.74}$$

Also the expression (B.72) has been achieved after using the relation

$$\mathfrak{f}_{ABC}\mathfrak{f}_{ABC} = 18\Psi^2 - 2\lambda\nabla_A\rho_A. \tag{B.75}$$

Finally we can get the 3-dimensional Einstein equation induced from the NC U(1) gauge fields

$$\begin{aligned} E_{AB} &= R_{AB} - \frac{1}{2}\delta_{AB}R \\ &= 8\pi G_3\big(T^{(M)}_{AB} + T^{(L)}_{AB}\big) \end{aligned} \tag{B.76}$$

where the Maxwell energy-momentum tensor and the Liouville energy-momentum tensor are, respectively, given by

$$T^{(M)}_{AB} = -\frac{1}{8\pi G_3\lambda^2}\left(\mathfrak{f}_{ACD}\mathfrak{f}_{BCD} - \frac{1}{4}\delta_{AB}\mathfrak{f}_{CDE}\mathfrak{f}_{CDE}\right) \tag{B.77}$$

$$T^{(L)}_{AB} = \frac{1}{16\pi G_3\lambda^2}\left(\frac{1}{2}\Big(\widetilde{\nabla}_A\rho_B + \widetilde{\nabla}_B\rho_A + \rho_A\rho_B\Big) - \delta_{AB}\left(\widetilde{\nabla}_C\rho_C + \frac{1}{2}(\rho_C\rho_C - 9\Psi^2)\right)\right) \tag{B.78}$$

where $\widetilde{\nabla}_A = \lambda\nabla_A$.

Following the exactly same strategy as the four dimensional case, one can identify $\widehat{T}^{(M)}_{AB}(\mathcal{A}_\theta)$ from eq. (B.77) getting the same form as eq. (B.44). Once again we get an exotic form of energy described by eq. (B.78) in addition to the usual Maxwell energy-momentum tensor. This energy density is also related to the gradient volume energy. (See section 3.2.) But the explicit form is different from the four dimensional one, eq. (B.38). This difference is due to the fact that the first term in eq. (B.15), which appears as the covariant derivative terms in eq. (B.78), is absent in four dimensions. An interesting thing in eq. (B.78) is that ρ_A behaves like a massive field whose mass is vanishing in flat spacetime since $\lambda = 1$ in that case. We further discuss in section 3.4 about the physical implications of the Liouville energy-momentum tensor.

In higher $D \geq 5$ dimensions, the calculation of the energy-momentum tensor from eq. (B.15) becomes more complicated. The 3-form field (B.47) contributes nontrivially to the energy-momentum tensor. We have not tried to find its concrete form. We hope to attack this problem in the near future.

JHEP05(2009)012

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JHEP05(2009)012